Analytic aspects in the evaluation of some multiple zeta values

Cezar Lupu[1]

[1]Department of Mathematics
University of Pittsburgh
Pittsburgh, PA, USA

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A quick overview of the Riemann zeta function.

The *Riemann zeta function* is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \operatorname{Re} s > 1.$$

Originally, Riemann zeta function was defined for real arguments. Also, Euler found another formula which relates the Riemann zeta function with prime numbrs, namely

$$\zeta(s) = \prod_p \frac{1}{\left(1 - \frac{1}{p^s}\right)},$$

where p runs through all primes $p = 2, 3, 5, \ldots$.

A quick overview of the Riemann zeta function.

Moreover, Riemann proved that the following $\zeta(s)$ satisfies the following integral representation formula:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{u^{s-1}}{e^u - 1} du, \operatorname{Re} s > 1,$$

where $\Gamma(s) = \displaystyle\int_0^\infty t^{s-1} e^{-t} dt$, $\operatorname{Re} s > 0$ is the Euler gamma function.

Also, another important fact is that one can extend $\zeta(s)$ from $\operatorname{Re} s > 1$ to $\operatorname{Re} s > 0$. By an easy computation one has

$$(1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^s},$$

and therefore we have

A quick overview of the Riemann function.

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^s}, \operatorname{Re} s > 0, s \neq 1.$$

It is well-known that ζ is analytic and it has an analytic continuation at $s = 1$. At $s = 1$ it has a simple pole with residue 1. We have

$$\lim_{s \to 1} (s-1)\, \zeta(s) = 1.$$

Let us remark that the alternating zeta function is called the *Dirichlet eta function* and it is defined as

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^s}, \operatorname{Re} s > 0, s \neq 1.$$

A quick overview of the Hurwitz zeta function.

Another important function is the *Hurwitz (generalized) zeta function* defined by

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \operatorname{Re} s > 1, a \neq 0, -1, -2, \ldots.$$

As the Riemann zeta function, Hurwitz zeta function is analytic over the whole complex plane except $s = 1$ where it has a simple pole. Also, from the two definitions, one has

$$\zeta(s) = \zeta(s, 1) = \frac{1}{2^s - 1} \zeta\left(s, \frac{1}{2}\right) = 1 + \zeta(s, 2).$$

A quick overview of the Hurwitz zeta function.

It can also be extended by analytic continuation to a meromorphic function defined for all complex numbers $s \neq 1$. At $s = 1$ it has a simple pole with residue 1. The constant term is given by

$$\lim_{s \to 1} \left(\zeta(s, a) - \frac{1}{s-1} \right) = -\frac{\Gamma'(a)}{\Gamma(a)} = -\Psi(a),$$

where Ψ is the digamma function. Also, the Hurwitz zeta function is related to the *polygamma function*,

$$\Psi_m(z) = (-1)^{m+1} m! \zeta(m+1, z).$$

What is known about the values of $\zeta(s)$ at integers?

- $\zeta(-2n) = 0$ for $n = 1, 2, \ldots$ (trivial zeros)
- $\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$; with $\zeta(-1) = -\frac{1}{12}$
- The values $\zeta(2n)$, for $n = 1, 2, \ldots$ have been found by Euler in 1740
- The values $\zeta(-2n+1)$, for $n = 1, 2, \ldots$ can be evaluated in terms of $\zeta(2n)$. In fact, we have

$$\zeta(-2n+1) = 2(2\pi)^{2n}(-1)^n(2n-1)!\zeta(2n).$$

- There is a mystery about $\zeta(2n+1)$ values!
- $\zeta(0) = -\frac{1}{2}$
- $\zeta(1)$ does not exist, but one has the following

$$\lim_{s\to 1}\left(\zeta(s) - \frac{1}{s-1}\right) = \gamma.$$

Another quick look at $\zeta(2n)$ and $\zeta(2n+1)$

In 1734, Euler produced a sensation when he discovered that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Later, the same Euler generalized the above formula,

$$\zeta(2n) = (-1)^{n+1} \cdot \frac{B_{2n} 2^{2n-1} \pi^{2n}}{(2n)!},$$

where the coefficients B_n are the so-called Bernoulli numbers and they satisfy

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n, |z| < 2\pi.$$

Another quick look at $\zeta(2n)$ and $\zeta(2n+1)$

An elementary but sleek proof of Euler's result was recently given in

E. De Amo, M. Diaz Carrillo, J. Hernandez-Sanchez, Another proof of Euler's formula for $\zeta(2k)$, *Proc. Amer. Math. Soc.* **139** (2011), 1441–1444.

The authors proved Euler's formula using the Taylor series for the tangent function and Fubini's theorem.
Unlike $\zeta(2n)$, the values $\zeta(2n+1)$ are still mysterious! One of the most important results was produced by Roger Apery in 1979, when he proved that $\zeta(3)$ is irrational by using the "fast converging" series representation

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3 \binom{2n}{n}}.$$

Another quick look at $\zeta(2n)$ and $\zeta(2n+1)$

Amazingly, there exist similar formulas for $\zeta(2)$ and $\zeta(4)$, namely

$$\zeta(2) = 3\sum_{n=1}^{\infty} \frac{1}{n^2\binom{2n}{n}}, \zeta(4) = \frac{36}{17}\sum_{n=1}^{\infty} \frac{1}{n^4\binom{2n}{n}}.$$

Recently, other substantial results were obtained. In 2002, K. Ball and T. Rivoal proved the following

K. Ball, T. Rivoal, Irrationalite d'une infinite de la fonction zetaaux entiers impairs, *Invent. Math.* **146** (2001), 193–207.

Theorem

There are infinitely many irrational values of the Riemann zeta function at odd positive integers. Moreover, if $N(n) = \#\{\text{irrational numbers among } \zeta(3), \zeta(5), \dots, \zeta(2n+1)\}$, *then* $N(n) \geq \frac{1}{2(1+\log 2)}\log n$ *for large* n.

Another quick look at $\zeta(2n)$ and $\zeta(2n+1)$

Other remarkable results in this direction are given by Rivoal (2001) and Zudilin (2001),

Theorem

At least four numbers $\zeta(5), \zeta(7), \dots, \zeta(21)$ *are irrational.*

and

W. Zudilin, One of the numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational, *Russ. Math. Surv.* **56** (2001), 193–206.

Theorem

At least one of the four numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ *is irrational.*

Another quick look at $\zeta(2n)$ and $\zeta(2n+1)$

The ultimate goal would be

Transcendence conjecture. *The numbers π, $\zeta(3)$, $\zeta(5)$, $\zeta(7)$, $\ldots$ are algebraically indepedendent. That is, for each integer $k \geq 0$ and each nonzero polynomial $P \in \mathbb{Z}[x_0, x_1, \ldots, x_k]$, we have $P(\pi, \zeta(3), \zeta(5), \ldots, \zeta(2k+1)) \neq 0$.*

Q: What do we know in this direction so far?

- In 1882, Lindemann proved that π is transcendental.
- From Euler's formula for $\zeta(2k)$ and previous result of Lindemann, it follows that $\zeta(2k)$ are transcendental.
- We do not know whether $\zeta(3)$ is transcendental or not.
- We do not know even if $\zeta(5)$ is irrational or not.

An introduction to the Multiple Zeta Values (MZV)

Multiple zeta values (Euler-Zagier sums) are the numbers defined by the following convergent series

$$\zeta(k_1, k_2, \dots, k_r) = \sum_{0<n_1<n_2<\dots<n_r} \frac{1}{n_1^{k_1} n_2^{k_2} \dots n_r^{k_r}},$$

where $k_1, k_2, \dots, k_r$ are positive integers with $k_r > 1$. Although it looks simple, it seems that these numbers have deep connections with Galois representation theory or they even appear in calculating Feynman integrals from quantum field theory. We call the above series a multiple zeta of weight k and depth r, where $k = k_1 + k_2 + \dots + k_r$. Obviously, $0 < r < k$ and there are $\binom{k-2}{r-1}$ multiple zeta values of given weight k and depth r.

An introduction to the Multiple Zeta Values (MZV)

Also, the MZVs can be rewritten as nested sums such as

$$\zeta(k_1, k_2, \ldots, k_r) = \sum_{n_r=1}^{\infty} \frac{1}{n_r^{k_r}} \cdots \sum_{n_2=1}^{n_3-1} \frac{1}{n_2^{k_2}} \sum_{n_1=1}^{n_2-1} \frac{1}{n_1^{k_1}}.$$

Moreover, due to Kontsevich, the multiple zeta values can be expressed as iterated integrals (or Drinfel'd integrals) defined by

$$\zeta(k_1, k_2, \ldots, k_r) = \int_{0<t_1<t_2<\ldots<t_k<1} \Omega_1\Omega_2\ldots\Omega_k = \int_0^1 \Omega_1\Omega_2\ldots\Omega_k,$$

where

$$\Omega_j = \begin{cases} \frac{dt_j}{1-t_j}, & j = 1, k_1+1, \ldots, k_1+k_2+\ldots+k_{r-1}+1 \\ \frac{dt_j}{t_j}, & otherwise \end{cases}.$$

Examples of integral representation of the MZVs

The following identity holds true:

$$\zeta(2) = \int_{0 \le t_1 \le t_2 \le 1} \frac{dt_1}{1-t_1} \frac{dt_2}{t_2} = \int_0^1 \left(\int_0^{t_2} \frac{dt_1}{1-t_1} \frac{1}{t_2} \right) dt_2.$$

Now, for $0 \le t_1 < 1$ we have the geometric series expansion, $\frac{1}{1-t_1} = \sum_{n \ge 1} t_1^{n-1}$, and thus

$$\int_0^{t_2} \frac{dt_1}{1-t_1} = \sum_{n \ge 1} \int_0^{t_2} t_1^{n-1} dt_1 = \sum_{n \ge 1} \frac{t_2^n}{n}.$$

Therefore, we get

$$\int_0^1 \frac{1}{t_2} \int_0^{t_2} \frac{dt_1}{1-t_1} dt_2 = \int_0^1 \sum_{n \ge 1} \frac{t_2^n}{n} \frac{dt_2}{t_2} = \sum_{n \ge 1} \frac{1}{n} \int_0^1 t_2^{n-1} dt_2 = \sum_{n \ge 1} \frac{1}{n^2}.$$

Examples of integral representation of the MZVs

Similarly, one has the following integral representation:

$$\zeta(1,2) = \int_{0 \le t_1 \le t_2 \le t_3 < 1} \frac{dt_1}{1-t_1} \frac{dt_2}{1-t_2} \frac{dt_3}{t_3}.$$

Some important evaluations of the MZV

- Euler, 1775:

$$\zeta(1,2) = \zeta(3).$$

- Hoffman, 1992:

$$\zeta(\{1\}^r, 2) = \zeta(r+2) \text{ and } \zeta(\{1,2\}^r) = \zeta(\{3\}^r).$$

- Hoffman & Zagier (independently), 1992:

$$\zeta(\{2\}^r) = \frac{\pi^{2r}}{(2r+1)!}.$$

- Zagier's conjecture (1992) and solved by Broadhurst (1998):

$$\zeta(\{1,3\}^r) = 4^{-r}\zeta(\{4\}^r) = \frac{2\pi^{4r}}{(4r+2)!}$$

Some important evaluations of the MZV

- Drinfel'd duality (1996):

$$\zeta(\{1\}^m, n+2) = \zeta(\{1\}^n, m+2).$$

- Granville (1997):

$$\sum_{k_1+k_2+\ldots k_r=m} \zeta(k_1, k_2, \ldots, k_r+1) = \zeta(m+1).$$

Some important evaluations of the MZV

- Zagier (2012):

$$\zeta(\{2\}^a, 3, \{2\}^b) = 2 \sum_{r=1}^{a+b+1} (-1)^r \cdot c^r_{a,b} \zeta(2r+1)\zeta(\{2\}_{a+b+1-r}),$$

where a, b are nonnegative integers and

$$c^r_{a,b} = \binom{2r}{2a+2} - \left(1 - \frac{1}{2^{2r}}\right)\binom{2r}{2b+1}.$$

The evaluation of Zagier's formula is important in showing that all multiple zeta values can be expressed as $\mathbb{Q}$-linear combinations of multiple zeta values of the same weight when all arguments are 2's and 3's.

Another proof of Hoffman-Zagier formula

Using Taylor series expansion of $\exp(a \arcsin x)$ and iterated integrals, one can derive the following formula:

$$\arcsin^{2r} x = (2r)! \sum_{n=1}^{\infty} \frac{H_r(n) 4^n}{n^2 \binom{2n}{n}} x^{2n}, |x| \leq 1$$

where $H_1(n) = \frac{1}{4}$ and

$$H_{r+1}(n) := \frac{1}{4} \sum_{n_1=1}^{n-1} \frac{1}{(2n_1)^2} \sum_{n_2=1}^{n_1-1} \frac{1}{(2n_2)^2} \cdots \sum_{n_r=1}^{n_{r-1}-1} \frac{1}{(2n_r)^2}.$$

Another proof of Hoffman-Zagier formula

We have the following

Theorem

$$\sum_{n=1}^{\infty} \frac{H_r(n)}{n^2} = \frac{\pi^{2r}}{2^{2r}(2r+1)!}.$$

The main ingredients in the proof of the above theorem are:

- $$\arcsin^{2r} x = (2r)! \sum_{n=1}^{\infty} \frac{H_r(n) 4^n}{n^2 \binom{2n}{n}} x^{2n}, |x| \leq 1.$$
- Wallis' integral formula:

$$\binom{2n}{n} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} (2\sin t)^{2n} dt$$

An evaluation of the Hoffman basis

We have the following

Theorem

$$\sum_{n=1}^{\infty} \frac{H_r(n)}{n^3} = -2\pi^2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+2r)(2n+2r+1)2^{2n}} \cdot \frac{\pi^{2r}}{2r+1}.$$

Thank you for your attention!!!