Reducing subspaces of the Dirichlet space

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  Bergman space:
  $L_a^2(\mathbb{D}) = \{f \in Hol(\mathbb{D}) : \|f\|_{L_a^2(\mathbb{D})}^2 = \int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty\}$

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- Dirichlet space: $D = \{f \in Hol(\mathbb{D}) : \int_{\mathbb{D}} |f'|^2 dA < \infty\}$
  norm: $\|f\|_D^2 = \|f\|_{H^2}^2 + \int_{\mathbb{D}} |f'|^2 dA$

Background on reducing subspaces

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- Reducing subspace: Let $\mathcal{H} = H^2, L^2_a$ or D, $T \in B(\mathcal{H})$. A closed subspace $\mathcal{M}$ is called a reducing subspace of T if $\mathcal{M}$ is invariant for both T and T^*

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- On H^2, the reducing subspaces of M_ϕ are in one-to-one correspondence with the closed subspaces of $H^2 \ominus \phi H^2$

- On L_a^2: when $n = 2$, in 1998, S. L. Sun, Y. J. Wang showed that M_ϕ has exact 2 minimal reducing subspaces on L_a^2. In 2000, Zhu proved this result using a different method; Zhu conjectured that for a finite Blaschke product ϕ of order n, there are exactly n distinct minimal reducing subspaces of M_ϕ on L_a^2.

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  In 2012, Douglas, Putinar and K. Wang proved the modified conjecture.

- Let $\mathcal{A}_\phi = \{M_\phi, M_\phi^*\}' = \{A \in B(L_a^2) : AM_\phi = M_\phi A, AM_\phi^* = M_\phi^* A\}$. Then $\mathcal{A}_\phi$ is a von Neumann algebra.

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- On D: when $n = 2$, under the norm $\| \cdot \|_1$, L. K. Zhao (2009) showed that M_ϕ is reducible if and only if ϕ is equivalent to z^2, i.e. $\phi = \varphi_\lambda(z^2), \lambda \in \mathbb{D}$, where $\|f\|_1^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'|^2 dA$.

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  When $n \geq 3$, it is unknown when M_ϕ is reducible on either Dirichlet space with the norm $\| \cdot \|_1$ or $\| \cdot \|_D$.

Main Results

Let $U : D \to L_a^2$, $Uf = (zf)'$, then U is a unitary operator.

Theorem 2.1 (L)

Let ϕ be a finite Blaschke product. If $\mathcal{M}$ is a reducing subspace of M_ϕ on D, then $U\mathcal{M} = (z\mathcal{M})'$ is a reducing subspace of M_ϕ on L_a^2.

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We say that two Blaschke products ϕ_1 and ϕ_2 are equivalent if there exist $\lambda \in \mathbb{D}$ such that $\phi_2 = \varphi_\lambda \circ \phi_1$.

Theorem 2.2 (L)

Let ϕ be a finite Blaschke product of order 3. Then M_ϕ is reducible on D if and only if ϕ is equivalent to z^3.

Theorem 2.3 (Gu-Xiao-L)

Let ϕ be a finite Blaschke product of order $n = 5$ or 7. Then M_ϕ is reducible on D if and only if ϕ is equivalent to z^n.

Let $E = \{\beta \in \mathbb{D} : \exists\ \alpha \in \mathbb{D}, \phi'(\alpha) = 0$ and $\phi(\alpha) = \phi(\beta)\}$, then E is a finite set, and $\phi^{-1} \circ \phi$ is an n-branched analytic function defined and arbitrarily continuable in $\mathbb{D} \setminus E$.

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For an open set $V \subseteq \mathbb{D}$, a local inverse of ϕ in V is a function ρ analytic in V which satisfies $\rho(V) \subseteq \mathbb{D}$ and $\phi(\rho(z)) = \phi(z)$ on V. Therefore there are n local inverses $\rho_0, \rho_1, \cdots, \rho_{n-1}$ for ϕ in $\mathbb{D} \setminus E$, i.e.

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$$\phi^{-1} \circ \phi = \{\rho_0(z), \rho_1(z), \cdots, \rho_{n-1}(z)\}.$$

We say that $\rho_i \sim \rho_j$ if there is a loop γ in $\mathbb{D} \backslash E$ such that ρ_i and ρ_j are analytic continuation of each other along γ. Then $\sim$ is an equivalence relation. Using this equivalence relation, we obtain a partition $\{G_1, G_2, \cdots, G_q\}$ for $\{\rho_0, \rho_1, \cdots, \rho_{n-1}\}$.

Define

$$\xi_i f(z) = \sum_{\rho \in G_i} f(\rho(z))\rho'(z) \quad z \in \mathbb{D} \setminus E, f \in L^2_a.$$

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Theorem 2.4 (Douglas-S. H. Sun-Zheng, 2011)

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Theorem 2.5 (Douglas-Putinar-K. Wang, 2012)

Let ϕ be a finite Blaschke product. Then The von Neumann algebra $\mathcal{A}_\phi$ is commutative of dimension q.

Let $\widetilde{\mathcal{A}_\phi} = \{M_\phi, M_\phi^*\}' \subset \mathrm{B}(D)$. Recall that $U : D \to L_a^2$, $Uf = (zf)'$ is a unitary operator.

If $\mathcal{M}$ is a reducing subspace of M_ϕ on D, then $U\mathcal{M} = (z\mathcal{M})'$ is a reducing subspace of M_ϕ on L_a^2.

Lemma

Let $T \in \widetilde{\mathcal{A}_\phi}, f \in D$. Then there are $a_1, \cdots, a_q \in \mathbb{C}$ such that

$$Tf(z) = \sum_{i=1}^{q} a_i \frac{F_i(z) - F_i(0)}{z}, \quad T^* f(z) = \sum_{i=1}^{q} \overline{a_i} \frac{H_i(z) - H_i(0)}{z}$$

where $F_i(z) = \sum_{\rho \in G_i} f(\rho(z))\rho(z)$, $H_i(z) = \sum_{\rho \in G_i^{-1}} f(\rho(z))\rho(z)$ and $G_i^{-1} = \{\rho : \rho^{-1} \in G_i\}$.

Let

$$\mathcal{L} = \text{span}\left\{(a_1, \cdots, a_q) : f \in D, \sum_{i=1}^{q} a_i F_i(0) = 0, \sum_{i=1}^{q} \overline{a_i} H_i(0) = 0\right\}.$$

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Theorem 2.7 (Gu-Xiao-L)

$\widetilde{\mathcal{A}_\phi}$ is a commutative von Neumann algebra, and $\dim \widetilde{\mathcal{A}_\phi} = \dim \mathcal{L}$.

Note that the family of local inverses $\{\rho_0, \cdots, \rho_{n-1}\}$ has a group-like property under composition near the boundary of $\mathbb{D}$. Write $j \in G_k$ if $\rho_j \in G_k$, then $\{G_1, G_2, \cdots, G_q\}$ is a partition of the additive group $\mathbb{Z}_n = \{0, 1, \cdots, n-1\}$.

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Necessary conditions for the partitions $\{G_1, G_2, \cdots, G_q\}$.

(A_1) One of $\{G_k\}$ is $\{0\}$ since $\rho_0(z) = z$.

(A_2) For each $G_j = \{j_1, \cdots, j_m\}$, there exists k such that

$$G_k = G_j^{-1} = \{n - j_1, \cdots, n - j_m\}.$$

(A_3) For any G_j, G_k, there are $G_{l_1}, \cdots, G_{l_m}$ such that

$$G_j + G_k = G_{l_1} \cup \cdots \cup G_{l_m} \quad \text{counting multiplicities on both sides.}$$

Let ϕ be a finite Blaschke product of order 5. We have the following possible partitions $\{G_1, G_2, \cdots, G_q\}$. By Corollary 8.4 of Douglas-Putinar and K. Wang, $q \neq 4$. We have the following cases.

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(i) If $q = 5$, then the partition is $\{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}\}$.

(ii) If $q = 3$, without loss of generality, suppose $G_1 = \{0\}$. Let $m = \min\{\#G_2, \#G_3\}$. By condition (A_3), m can not be 1. Thus $m = 2$, then $\#G_2 = \#G_3 = 2$, and there are essentially three cases.

(a) $G_2 = \{1, 2\}, G_3 = \{3, 4\}$;

(b) $G_2 = \{1, 3\}, G_3 = \{2, 4\}$;

(c) $G_2 = \{1, 4\}, G_3 = \{2, 3\}$.

Case (a) doesn't satisfy condition (A_3), since $G_2 + G_2 = \{2, 3, 3, 4\}$. Similarly, case (b) doesn't satisfy condition (A_3). So we have the possible partition (c).

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(iii) If $q = 2$, then the partition is $\{\{0\}, \{1, 2, 3, 4\}\}$.

Therefore when $n = 5$, the possible partitions are

$$\begin{cases} \{\{0\},\{1\},\{2\},\{3\},\{4\}\},\ \{\{0\},\{1,4\},\{2,3\}\}; \\ \{\{0\},\{1,2,3,4\}\}. \end{cases}$$

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Theorem (Gu-Xiao-L)

Let ϕ be a finite Blaschke product of order 5. Then one of the following holds:

(a) *If ϕ is equivalent to $\varphi_\alpha^5, \alpha \in \mathbb{D}$, then the partition is $\{\{0\},\{1\},\{2\},\{3\},\{4\}\}$;*

(b) *If ϕ is equivalent to $(z^2\varphi_\alpha^2\varphi_\beta)\circ\varphi_\gamma, \alpha, \beta \in \mathbb{D}\backslash\{0\}, \gamma \in \mathbb{D}, \alpha/\beta \in \mathbb{R}, \varphi_\beta(\alpha) = \frac{\alpha^2}{\beta}$, then the partition is $\{\{0\},\{1,4\},\{2,3\}\}$;*

(c) *If ϕ is not equivalent to any of the functions in (a) and (b), then the partition is $\{\{0\},\{1,2,3,4\}\}$.*

Thank You!