Toeplitz Operators on Framed Hilbert Spaces

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Toy example

Let $\{e_n\}_{n=1}^{\infty}$ be some orthonormal basis for a separable Hilbert space H and $\{a_n\}_{n=1}^{\infty}$ is some positive sequence of numbers.

Consider the operator $T : Span\{e_n\} \to H$ given by

$$Tf = \sum_n a_n \langle f | e_n \rangle e_n.$$

Notice that T is a positive symmetric operator with eigen-pairs (a_n, e_n).

This is the simplest example of Toeplitz operator.

Toy example

The matrix of T under the orthonormal eigen-basis $\{e_n\}_{n=1}^{\infty}$ is given by

$$[T]= \begin{pmatrix} \langle Te_1|e_1\rangle & 0 & \cdots & 0 & \cdots \\ 0 & \langle Te_2|e_2\rangle & \cdots & 0 & \cdots \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & \cdots & \langle Te_n|e_n\rangle & \cdots \\ \vdots & \vdots & & \vdots & \end{pmatrix}$$

The diagonal terms which are the eigenvalues

$$\langle Te_n|e_n\rangle = a_n$$

give a characterization of the boundedness, compactness and Schatten class membership of T.

Toy example

Theorem

(i) T is bounded iff $\{a_n\}_{n=1}^{\infty} = \{\langle Te_n | e_n \rangle\}$ *is bounded.*

(ii)T is compact iff $\{a_n\}_{n=1}^{\infty} \in c_0$ *i.e.* $a_n \to 0$ *as* $n \to \infty$.

(iii)$T \in S_p$ *(Schatten class* $p \geq 1$*) iff* $\{a_n\}_{n=1}^{\infty} \in l^p(\mathbb{N})$.

Toy example

Our goal is to generalize the previous theorem, in the case when the orthonormal basis $\{e_n\}_{n=1}^{\infty}$ is replaced by a much more general "basis" called generalized Parseval frame $\{k_x\}_{x\in X}$.

The general class of operators that we study has the following form $Tf = \int_X a(x) \langle f|k_x\rangle k_x d\lambda(x)$ where λ is some measure on X and $a : X \to \mathbb{R}^+$

Notice in the toy example $a_n = \langle Te_n|e_n\rangle$. In general, $a(x) \neq \langle Tk_x|k_x\rangle$, but still we can tell operator-theoretic properties of T by judging from $\langle Tk_x|k_x\rangle$.

definition of Toeplitz operators

We can replace $a(x)d\lambda(x)$ with a general measure $d\mu(x)$ and get a slightly large class of operators $T_\mu f = \int_X \langle f|k_x\rangle\, k_x d\mu(x)$. To define T_μ rigorously, we take μ to be the positive Borel measure on X such that for any $x \in X$ we have

$$\int_X |\langle k_x|k_y\rangle|\, d\mu(y) < \infty$$

For any $x \in X$, we define $l_x : H \to \mathbb{C}$ by

$$l_x(f) = \int\limits_X \langle k_x|k_y\rangle \langle k_y|f\rangle\, d\mu(y)$$

We can prove l_x is well-defined bounded anti-linear complex valued functional on H.

definition of Toeplitz operators

By Riesz representation theorem, there exists $h_x \in H$ such that for any $f \in H$:

$$\langle h_x | f \rangle = l_x(f)$$

Now we can densely define T_μ called Toeplitz Operator by

$$\begin{aligned} T_\mu : Span\{k_x\} &\longrightarrow H \\ k_x &\longrightarrow h_x \end{aligned}$$

Notice for any $x \in X$ $f \in H$

$$\langle T_\mu k_x | f \rangle = \int_X \langle k_x | k_y \rangle \langle k_y | f \rangle \, d\mu(y)$$

so we denote $T_\mu k_x$ by $\int_X \langle k_x | k_y \rangle \, k_y d\mu(y)$

Basic definitions and assumptions

A framed Hilbert space is a triple (H, X, k) such that

1. H is a Hilbert Space

2. (X, d, λ) is metric measure space, d is the metric on X, λ is the Borel measure with respect to the metric d.

3. k is a generalized Parseval Frame which is a continuous function given by

$$\begin{aligned} k : X &\longrightarrow H \\ x &\longrightarrow k_x \end{aligned}$$

and satisfies

$$\|f\|^2 = \int_X |\langle f|k_x\rangle|^2 d\lambda(x), \ \forall f \in H$$

Basic definitions and assumptions

If $\{k_x\}$ is a generalized Parseval frame, we can write

$$f = \int_X \langle f|k_x\rangle k_x d\lambda(x), \text{ for all } f \in H,$$

which means

$$\langle f|g\rangle = \int_X \langle f|k_x\rangle \langle k_x|g\rangle d\lambda(x) \text{ for all } f, g \in H.$$

Specifically, when $X = \mathbb{N}$ with the counting measure λ, k will be the usual Parseval frame. And every orthonormal basis is a Parseval frame.

Basic definitions and assumptions

We will assume the metric measure space (X, d, λ) has a finite asymptotic dimension N. That means there is N, such that for any $r > 0$ there exists the collection of Borel sets $\{F_n\}_{n=1}^{\infty}$ which satisfies

1. $X = \bigcup\limits_{n=1}^{\infty} F_n$
2. $F_n \bigcap F_m = \emptyset$, $n \neq m$

3. Any $x \in X$ is contained in at most N sets of $\{G_n\}_{n=1}^{\infty}$, where $G_n := \{x \in X : d(x, F_n) \leqslant r\}$ is the r-neighborhood of F_n.

4. $diam(F_n) \leqslant C_r$, $\forall n \in \mathbb{N}$
5. There exist positive constants A_r and B_r, such that for any $n \in \mathbb{N}$

$$0 < A_r \leqslant \lambda(F_n) \leqslant \lambda(G_n) \leqslant B_r < \infty$$

Basic definitions and assumptions

Recall every doubling metric measure space (X, d, λ) has a finite asymptotic dimension.

And every Gromov hyperbolic geodesic metric space (X, d) with bounded growth has a finite asymptotic dimension.

Basic definitions and assumptions

Furthermore, we will make the following assumptions for the generalized Parseval frame $\{k_x\}_{x \in X}$.

1. $\|k_x\| = 1, \forall x \in X$

2. (Mean value property) For any $r > 0$, there exists $C_r > 0$, such that for any $x \in X, f \in H$

$$|\langle f|k_x\rangle|^2 \leqslant C_r \int\limits_{B(x,r)} |\langle f|k_y\rangle|^2 d\lambda(y)$$

Basic definitions and assumptions

3. There exists an increasing function $\phi : [0, \infty) \longrightarrow [1, \infty)$ satisfying $\phi(0) = 1$ and $\phi(r) \to \infty$ as $r \to \infty$, such that

$$0 < c \leq |\langle k_x | k_y \rangle| \phi(d(x, y)) \leq C < \infty$$

where c and C are positive constants.

Examples of framed Hilbert spaces

Bergman space $\mathcal{A}^2$

- $X = \mathbb{D}$

- $\mathcal{A}^2 := \{f : \mathbb{D} \to \mathbb{C} | f \text{ is holomorphic on } \mathbb{D} \text{ and } \int_{\mathbb{D}} |f(z)|^2 dA_\alpha(z) < \infty\}$

where $dA_\alpha(z) := \frac{\alpha+1}{\pi}(1 - |z|^2)^\alpha dxdy$, $(-1 < \alpha < \infty)$

- $d\lambda$ is the invariant hyperbolic measure on $\mathbb{D}$

- the metric d is the Bergman metric

- the generalized Parseval frame $\{k_z\}_{z \in \mathbb{D}}$ consists of normalized reproducing kernels

Examples of framed Hilbert spaces

Fock space $\mathcal{F}^2$

- $X = \mathbb{C}$

- $\mathcal{F}^2 := \{f : \mathbb{C} \to \mathbb{C} | f \text{ is entire and } \int\limits_{\mathbb{C}} |f(z)|^2 dA_\alpha(z) < \infty\}$

where $dA_\alpha(z) := \frac{\alpha}{\pi} e^{-\alpha|z|^2} dxdy$, $(\alpha > 0)$

- $d\lambda$ is the Lebesgue measure on $\mathbb{C}$

- the metric d is the Euclidian distance

- the generalized Parseval frame $\{k_z\}_{z\in\mathbb{C}}$ consists of normalized reproducing kernels

Examples of framed Hilbert spaces

Paley-Wiener space $\mathcal{PW}^2$

- $X = \mathbb{R}$

- $\mathcal{PW}^2 := \{f : \mathbb{C} \to \mathbb{C} | f \text{ is entire of exponential type } \leq \alpha, \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty\}, (\alpha > 0)$

- $d\lambda$ is the Lebesgue measure on $\mathbb{R}$

- the metric d is the Euclidian distance

- the generalized Parseval frame $\{k_x\}_{x\in\mathbb{R}}$ consists of normalized reproducing kernels

Basic properties of Toeplitz operators

Recall $T_\mu f = \int_X \langle f | k_x \rangle k_x d\mu(x)$, we will be interested in when T_μ

i. is bounded

ii. is compact

iii. belong to S_p (Schatten Class S_p)

We will characterize boundedness, compactness and S_p membership of T_μ in terms of the Berezin transform of μ.

Berezin transform

Definition

The function $\tilde{\mu} : X \to \mathbb{C}$ given by $\tilde{\mu}(x) = \langle T_\mu k_x | k_x \rangle$ is called the Berezin transform of μ.

It is easy to see that

$$\tilde{\mu}(x) = \int_X |\langle k_x | k_y \rangle|^2 d\mu(y), \quad x \in X$$

Boundedness

Theorem

Let μ be a positive Borel measure on X that satisfies

$$\int_X |\langle k_x | k_y \rangle| d\mu(y) < \infty, \quad \textit{for all } x \in X$$

Then

(i) T_μ is bounded iff $\tilde{\mu}$ is bounded.

(ii) T_μ is compact iff $\tilde{\mu} \in C_0(X)$.

(iii) $T_\mu \in S_p$ (Schatten class $p \geq 1$) iff $\tilde{\mu} \in L^p(d\lambda)$.

Sketch of the proof

I will show the boundedness of $\tilde{\mu}$ implies the boundedness of T_μ. To prove this, I will follow two steps:

1. To show $\mu(F_n) \leq C$ for any $n \in \mathbb{N}$;
2. To show $|\langle T_\mu f | f \rangle| \leq C \|f\|^2$;

Proof.

First let $r > 0$, since X has N asymptotic dimension, we have for any $n \in \mathbb{N}$

$$\begin{aligned}\mu(F_n) &= \int_{F_n} 1 d\mu(y) \\ &\leq \int_{F_n} \frac{|\langle k_{x_n}|k_y\rangle|^2}{D_r^2} d\mu(y) \quad (x_n \text{ is some point in } F_n) \\ &\leq \frac{1}{D_r^2}\int_X |\langle k_{x_n}|k_y\rangle|^2 d\mu(y) \\ &= \frac{1}{D_r^2}\tilde{\mu}(x_n) \leq C.\end{aligned}$$

□

Sketch of the proof

$$
\begin{aligned}
|\langle T_\mu f|f\rangle| &= \int_X |\langle f|k_x\rangle|^2 d\mu(x) \\
&= \sum_n \int_{F_n} |\langle f|k_x\rangle|^2 d\mu(x) \\
&\leq \sum_n \int_{F_n} C_1^r \int_{B(x,r)} |\langle f|k_y\rangle|^2 d\lambda(y) d\mu(x) \\
&\leq C_1^r \sum_n \int_{F_n} \int_{G_n} |\langle f|k_y\rangle|^2 d\lambda(y) d\mu(x) \\
&= C_1^r \sum_n \mu(F_n) \int_{G_n} |\langle f|k_y\rangle|^2 d\lambda(y) \\
&\leq C_1^r C_2^r N \int_X |\langle f|k_y\rangle|^2 d\lambda(y) \\
&= C\|f\|^2.
\end{aligned}
$$

Thank you.

Appendix

Let o be some fixed point in X (basepoint).

For $x, y \in X$, their Gromov product $(x|y)$ is defined by $(x|y) = (d(x,o) + d(y,o) - d(x,y))/2$.

The metric space (X, d) is said to be Gromov hyperbolic if there exists $\delta > 0$ such that $(x|z) \geq \min\{(x|y), (y|z)\} - \delta$, for all $x, y, z \in X$.

Appendix

A metric space (X, d) is said to be of bounded growth if for each $r > 0$ there exists M_r such that for every $R > 0$ each ball of radius $R + r$ in X can be covered by at most M_r balls of radius R.