D_K spaces and Carleson measures

Dongxing Li

Joint with Hasi Wulan and Ruhan Zhao

The College at Brockport, State University of New York,
Mathematics Department
March 17, 2017

Notations

Let $\mathbb{D}$ denote the unit disc and $\mathrm{Hol}(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$. The Dirichlet space, denote by $\mathcal{D}$, consists of all $f \in \mathrm{Hol}(\mathbb{D})$ such as $D(f) := \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty$.
Let $K : [0, \infty) \mapsto [0, \infty)$ be right continuous and nondecreasing. We say a $f \in \mathrm{Hol}(\mathbb{D})$ belongs to the space D_K if

$$||f||_{D_K}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 K(1 - |z|^2) dA(z) < \infty. \quad (1)$$

If $K = t^s$, $0 < s < \infty$, it is simply denoted as D_s.

Background: Carleson measure

We say that a positive Borel measure μ on $\mathbb{D}$ is a Carleson measure if the embedding operator from the Hardy space H^p into $L^p(d\mu)$ is bounded, that is

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \leqslant C\|f\|_{H^p}^p \tag{2}$$

holds for all $f \in H^p$. It is well known that μ is a Carleson measure if and only if there exists a constant $C > 0$ such that

$$\mu(S(I)) \leqslant C|I| \tag{3}$$

for each arc $I \subset \partial\mathbb{D}$. Here
$S(I) = \left\{z \in \mathbb{D} : 1 - |I| < |z| < 1, |\theta - \arg z| < \frac{|I|}{2}\right\}$ is the so-called Carleson box, where θ is the middle point of I.

Remark.

We may extend the notion of the Carleson measure by replacing the right-hand side of (2) by the norm or the semi-norm of some other function spaces such as the Bergman space, the Bloch space, BMOA, etc.

Geometrically, giving an increasing function $\phi : (0, 2\pi) \mapsto (0, \infty)$, the classical Carleson one box condition $\mu(S(I)) = O(|I|)$ can be generalized as $\mu(S(I)) = O(\phi(|I|))$. For example, if $s > 0$ by taking $\phi(x) = x^s$ we may generalize the classical Carleson measure as follows: μ is an s-Carleson measure on $\mathbb{D}$ if $\mu(S(I)) = O(|I|^s)$.

We say that a positive Borel measure μ on $\mathbb{D}$ is a Carleson measure for D_K if there is a positive constant C such that

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \leqslant C\|f\|_{D_K}^2. \tag{4}$$

Especially, if $K(t) = t^s$, then we say μ is a Carleson measure for D_s.

In 1980, Stegenga [5] obtained that for $s \geqslant 1$, μ is an s-Carleson measure if and only if μ is a Carleson measure for D_s.

Motivation

Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and $g(z)=\sum_{n=0}^{\infty}b_nz^n$. The Hadamard product of f and g is defined as $f*g(z)=\sum_{n=0}^{\infty}a_nb_nz^n$.
Aulaskari, Girela and Wulan, (2000),[1] characterized the classical Carleson measure by using $f*g$. Namely, they obtained the following result. For $0<s<\infty$, a positive Borel measure μ on $\mathbb{D}$ is a classical Carleson measure if and only if there exists a positive constant C such that

$$\int_{\mathbb{D}}|f*g(z)|^2d\mu(z)\leqslant C\|f\|_{D_s}^2$$

for all $f\in D_s$, where $g(z)=1+\sum_{n=1}^{\infty}n^{\frac{1-s}{2}}z^n$. Our goal is to extend their result to the space D_K and to find out the concrete representation of $g(z)$.

Main results

Theorem 1. A positive Borel measure μ on $\mathbb{D}$ is an s-Carleson measure $(s \geqslant 1)$ if and only if there exists a constant $C > 0$ such that

$$\int_{\mathbb{D}} |f * g(z)|^2 d\mu(z) \leqslant C\|f\|_{D_K}^2$$

for all $f \in D_K$, where $g(z) = 1 + \sum_{n=1}^{\infty} \sqrt{n^s K(\frac{1}{n})} z^n$.

Theorem 2. A positive Borel measure μ on $\mathbb{D}$ is an s-Carleson measure *($s > 1$)* if and only if there exists a constant $C > 0$ such that

$$\int_{\mathbb{D}} |(f * g)'(z)|^2 d\mu(z) \leqslant C\|f\|_{D_K}^2$$

for all $f \in D_K$, where $g(z) = 1 + \sum_{n=1}^{\infty} \sqrt{n^{s-2}K(\frac{1}{n})}z^n$.

Idea of proof.

We only talk about the Theorem 1 as an example. The proof of sufficiency consists of two steps.

Step 1: Proving that $f \in D_K$ if and only if $f * g \in D_s$, and

$$\|f * g\|_{D_s} \leq C\|f\|_{D_K},$$

where $g(z) = 1 + \sum_{n=1}^{\infty} \sqrt{n^s K(\frac{1}{n})} z^n$.

Step 2. Replacing f by $f * g$ in the following definition of s-Carleson measure:

$$\int_{\mathbb{D}} |f|^2 d\mu(z) \leqslant C\|f\|_{D_s}^2,$$

and then combining with the above inequality, we arrive at the desired result.

Necessary: We only need to choose a proper test function f_a (which depends on $a \in \mathbb{D}$ and the function K) and use the fact that μ is s-Carleson measure if and only if

$$\sup_{a\in\mathbb{D}} \int_{S(a)} \frac{d\mu(z)}{|1-\bar{a}z|^s} < \infty,$$

where
$S(a) := \{z \in \mathbb{D} : |a| \leqslant |z| < 1; |\arg\frac{z}{a}| < \pi(1-|a|)\}, \quad a \neq 0$, is a so-called Carleson window.

Recently, El-Fallah, Kellay and Mashreghi (2015) obtained the following one box sufficient condition for the Carleson measure for the Dirichlet space $\mathcal{D}$: A finite positive Borel measure μ on $\mathbb{D}$ is a Carleson measure for $\mathcal{D}$ if

$$\mu(S(I)) = O(\phi(|I|)),$$

where $\phi : (0, 2\pi) \mapsto (0, \infty)$ is an increasing function such that $\int_0^{2\pi} \frac{\phi(x)}{x} dx < 1$.

Here we generalize the above result as follows.

Theorem 4. Let μ be a finite positive Borel measure on $\mathbb{D}$ satisfying $\mu(S(I)) = O(\phi(|I|))$, where $\phi : (0, 2\pi) \mapsto (0, \infty)$ is an increasing function such that $\int_0^{2\pi} \phi(x)\varphi'(x)dx < 1$. Then μ is a Carleson measure for D_K, where $\varphi(x) = \sum_{k=1}^{\infty} \frac{(1-x)^n}{\kappa_n}$, $\kappa_n = n \int_0^1 K(1 - t^{\frac{1}{n}})dt$.

The key of proof is the following dual formulation of the notion of Carleson measure (by Arcozzi, Rochberg, and Sawyer (2008) [2]): A finite positive measure μ on $\mathbb{D}$ is a Carleson measure for a Hilbert space H if

$$\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |k(z,w)| d\mu(z) < \infty, \tag{5}$$

where $k(z,w)$ is the reproducing kernel of H.
Our proof uses this result and the fact that the reproducing kernel of D_K is

$$k_{D_K}(z,w) = 1 + \sum_{n=1}^{\infty} \frac{(z\overline{w})^n}{\kappa_n}.$$

References

- R. Aulaskari, D. Girela and Hasi Wulan, *Q_p spaces, Hadamard products and Carleson measures,* Mathematical Report, 2000.
- Nicola Arcozzi, Richard Rochberg, and Eric Sawyer, C*Carleson measures for the Drury-Arveson Hardy space and other Besov-Sobolev spaces on complex balls,* Adv. Math. 218 (2008), no. 4, 1107-1180.
- L. Carleson, *Interpolations by bounded analytic functions and the corona problem,* Ann. of Math. (2) 76(1962), 547-559.
- El-Fallah O, Kellay K, Mashreghi J, et al. *One-box conditions for Carleson measures for the Dirichlet space,* Proceedings of the American Mathematical Society, 2015, 143(2): 679-684.
- D. A. Stegenga, *Multipliers of the Dirichlet space,* Illinois J. Math. 24(1980),113-139.