Algebraic properties of *m*-isometric commuting tuples

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SEAM 2017
University of Tennessee
March 17, 2017

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The Dirichlet shift $\mathcal{D}$ is a (non-isometric) 2-isometry:

$$\mathcal{D} e_n = \sqrt{\frac{n+2}{n+1}}\, e_{n+1} \quad \text{for } n \geq 0.$$

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- My approach may be used in the multivariate setting as well.

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- 1-**isometric** tuples are **spherical isometries**:

$$T_1^* T_1 + \cdots + T_d^* T_d = I$$

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$$K(z, \lambda) = \frac{1}{(1 - \langle z, \lambda \rangle)^a}.$$

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- Let $M_z = (M_{z_1}, \ldots, M_{z_d})$ be the commuting tuple of multiplication operators. We consider M_z as acting on $\mathcal{K}_{a,d}$.

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Theorem (Gleason-Richter (2006))

If a and d are positive integers and $d \geq a$, then M_z is a $(d - a + 1)$-isometric tuple on $\mathcal{K}_{a,d}$.

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If a and d are positive integers and $d \geq a$, then M_z is a $(d-a+1)$-isometric tuple on $\mathcal{K}_{a,d}$.

- In particular, the d-shift M_z on the Drury-Arveson space $H^2_d = \mathcal{K}_{1,d}$ is d-isometric.

Theorem (Richter-Sundberg (2011))

(a) If $\lambda = (\lambda_1, \ldots, \lambda_d) \in \partial\mathbb{B}_d$ *and* $V_j : \mathbb{C}^m \to \mathbb{C}^n$ *such that* $\bar{\lambda}_1 V_1 + \cdots + \bar{\lambda}_d V_d = 0$*, then* $S = (S_1, \ldots, S_d)$ *with*

$$S_j = \begin{pmatrix} \lambda_j I_n & V_j \\ 0 & \lambda_j I_m \end{pmatrix} : \mathbb{C}^{n+m} \to \mathbb{C}^{n+m}$$

is 2*-isometric.*

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(b) If T *is a 2-isometric commuting tuple on a finite dimensional space, then*

$$T = U \oplus S,$$

where U *is a spherical isometry and* S *is a direct sum of the above tuples.*

Products of m-isometric tuples

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Suppose $A = (A_1, \ldots, A_d)$ is m-isometric; $B = (B_1, \ldots, B_k)$ is n-isometric; and $A_j B_\ell = B_\ell A_j$ for all j, ℓ. Then $A * B$ is $(m+n-1)$-isometric, where

$$A * B = (d\,k)\text{-tuple consisting of all products } A_j B_\ell.$$

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In general, taking a sub-tuple is not enough.

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- Recall that if A is an isometry; $N^s = 0$; and $AN = NA$, then $A + N$ is $(2s-1)$-isometric.

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- If $N = (N_1, \ldots, N_d)$ is a commuting tuple, we say that N is s-**nilpotent** if

$$N^\alpha = N_1^{\alpha_1} \cdots N_d^{\alpha_d} = 0$$

for all $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $|\alpha| = \alpha_1 + \cdots + \alpha_d = s$.

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Suppose $A = (A_1, \dots, A_d)$ is m-isometric; $N = (N_1, \dots, N_d)$ is s-nilpotent; and $A_j N_\ell = N_\ell A_j$ for all j, ℓ. Then $A + N = (A_1 + N_1, \dots, A_d + N_d)$ is $(m + 2s - 2)$-isometric.

Transformations of m-isometric tuples

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- This generalizes to all automorphisms of the unit ball:

Theorem

Let $\varphi : \mathbb{B}_d \to \mathbb{B}_d$ be an automorphism. Let $A = (A_1, \ldots, A_d)$ be m-isometric, so $\varphi(A)$ is defined. Then $\varphi(A)$ is m-isometric as well.

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- Our approach makes use of the **hereditary calculus** (termed and studied by J. Agler).
- Given a commuting tuple $T = (T_1, \ldots, T_d)$. We define and extend by linearity the map

$$\begin{aligned} \mathbb{C}[\bar{z}, z] = \mathbb{C}[\bar{z}_1, \ldots, \bar{z}_d, z_1, \ldots, z_d] &\to \mathcal{B}(H) \\ \bar{z}^{\beta} z^{\alpha} &\mapsto (T^{\beta})^* T^{\alpha} \end{aligned}$$

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- Several authors consider non-commutative variables on the left side.
- However, for us, it is more convenient that the left side is **commutative**.

Our approach

- Let $p(z) = (|z|^2 - 1)^m = (|z_1|^2 + \cdots + |z_d|^2 - 1)^m$. Then T is m-isometric if and only if T is a **hereditary root of** p, that is,

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$$Q(\{\zeta_{j,\ell}\}) = \Big((\sum_{j,\ell} |\zeta_{j,\ell}|^2) - 1 \Big)^{m+n-1}$$

We need to show that $A * B = \{A_j B_\ell\}$ is a hereditary root of Q.

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- The binomial expansion gives

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- It then follows that $Q(\{A_j B_\ell\}^*, \{A_j B_\ell\}) = 0$, as required.

THANK YOU!