'Twisted Duality' for the Clifford von Neumann Algebra

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Jeramiah Hocutt
University of Florida

Outline

- Give background on Clifford algebras
- Define 'twisted duality' in purely algebraic terms
- Define enveloping Clifford von Neumann algebra
- Illustrate 'twisted duality' in the Clifford von Neumann algebra

Clifford Algebra

Let V be a real inner product space, with inner product $(\cdot|\cdot)$. A Clifford algebra of V is a Clifford map $\epsilon : V \to C(V)$ with

$$\forall v \in V, \qquad \epsilon(v)^2 = (v|v)\,\mathbf{1}$$

which satisfies the universal mapping property.

For each Clifford map $\phi : V \to A$ there exists a unique algebra map $\Phi : C(V) \to A$ such that the following diagram commutes:

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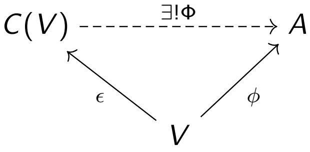

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- γ splits $C(V)$ into two pieces:

$$\begin{aligned} C_+(V) &= \{v \in C(V) |\ \gamma(v) = v\} \quad \text{(even)} \\ C_-(V) &= \{v \in C(V) |\ \gamma(v) = -v\} \quad \text{(odd)} \end{aligned}$$

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- $C(V) = C_+(V) \oplus C_-(V)$.

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- $C(Z)$ is the Clifford subalgebra of $C(V)$ generated by Z.
- The graded commutant of $C(Z)$ is

$$C(Z)^{\gamma} = C_{+}(Z)^{\gamma} \oplus C_{-}(Z)^{\gamma}$$

where

$$C_{+}(Z)^{\gamma} = \{a \in C_{+}(V) | \ (\forall b \in C(Z)) \ ba = ab\}$$

$$C_{-}(Z)^{\gamma} = \{a \in C_{-}(V) | \ (\forall b \in C(Z)) \ ba = \gamma(a)b\}$$

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Theorem (Robinson)

$$C(Z)^{\gamma} = C(Z^{\perp}) \qquad [1]$$

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Clifford Hilbert Space

- There exists a γ-invariant linear functional $\tau : C(V) \to \mathbb{C}$ (trace)
- Give $C(V)$ an I.P. defined by $\langle a|b\rangle_\tau = \tau(a^* b)$
- Complete $C(V)$ with respect to $\langle\cdot|\cdot\rangle_\tau$ to obtain the Hilbert space $\mathbb{H}[V]$.

von Neumann Algebra

- The left regular representation (LRR) $\lambda : C(V) \to \operatorname{End} C(V) : a \mapsto \lambda(a)$ where

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- We could also define

$$\mathcal{A}[V] = \overline{\lambda(C(V))}^{w} = \overline{\lambda(C(V))}^{s}.$$

von Neumann Algebra Structure

- γ extends via LRR to $\Gamma \in B(\mathbb{H}[V])$ by

$$\Gamma(a) = \begin{cases} a, & a \in \mathbb{H}_+ \\ -a, & a \in \mathbb{H}_- \end{cases}$$

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- Γ splits $\mathcal{A}[V]$ as before

$$\mathcal{A}[V] = \mathcal{A}_+[V] \oplus \mathcal{A}_-[V]$$

Graded Commutant

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- Graded Commutant:

$$\mathcal{A}[Z]^{\gamma} = \mathcal{A}_{+}[Z]^{\gamma} \oplus \mathcal{A}_{-}[Z]^{\gamma}$$

OR

$$\mathcal{A}[Z]^{\gamma} = \{T \in \mathcal{A}[V] \mid (z \in Z) \;\; \lambda(z)T = \Gamma T \Gamma \lambda(z)\}$$

Twisted Duality (Again)

For V a real inner product space and $Z \leq V$ a subspace, we have

Theorem

$$\mathcal{A}[Z]^{\gamma} = \mathcal{A}\left[Z^{\perp}\right].$$

Proof

One direction ($\supseteq$) is immediate from the (linearized) Clifford relations.
Some tools used in the proof:

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Some tools used in the proof:

- For $z \in Z$ define a conditional expectation

$$E_z(a) = \frac{1}{2}\left(a + z\gamma(a)z\right)$$

- E_z projects $C(V)$ onto $C(z^\perp)$
- $E_z(a) = a$ for all $a \in C(z)^\gamma$
- For any two $v, w \in Z$, $E_w E_v = E_v E_w$

Proof

Extend E_z to $\mathcal{A}[V]$ via LRR λ

$$\mathbb{E}_z(T) = \frac{1}{2}\left(T + \lambda(z)\Gamma T \Gamma \lambda(z)\right)$$

This has the same properties as the original E_z.

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For finite dimensional $Z \leq V$ with basis $\{z_1, \ldots, z_n\}$, define

$$\mathbb{E}_Z(T) = (\mathbb{E}_{z_n} \circ \cdots \circ \mathbb{E}_{z_1})(T)$$

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$$\mathbb{E}_Z(T) = (\mathbb{E}_{z_n} \circ \cdots \circ \mathbb{E}_{z_1})(T)$$

Since $\mathbb{E}_Z$ maps $\mathcal{A}[V]$ onto $\mathcal{A}\left[Z^{\perp}\right]$ and fixes elements of $\mathcal{A}[Z]^{\gamma}$, 'twisted duality' holds for finite dimensional Z.

Proof

To pass to infinite dimensional Z, we need more.

- Given a finite von Neumann algebra N with a fixed faithful trace, define $\langle \cdot | \cdot \rangle_\tau$ as before and complete to get $\mathbb{H}_N$. Embed $N \hookrightarrow \mathbb{H}_N : x \mapsto x1$ (τ faithful).

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- For B a von Neumann subalgebra of N, take $\mathbb{H}_B$ to be the closure of B in $\mathbb{H}_N$. We have the orthogonal projection $e_B : \mathbb{H}_N \to \mathbb{H}_B$.
- The restriction of e_B to N gives a trace preserving conditional expectation $\mathbb{E}_B$, i.e.

$$\mathbb{E}_B = e_B|_N : N \to B$$

  with $\mathbb{E}_B(a)1 = e_B(a1)$.

[2]

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Obtain a trace preserving conditional expectation $\mathbb{E}_Z : \mathcal{A}[V] \to \mathcal{A}\left[Z^{\perp}\right]$ by $\mathbb{E}_Z = e_Z|_{\mathcal{A}[V]}$.

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Obtain a trace preserving conditional expectation $\mathbb{E}_Z : \mathcal{A}[V] \to \mathcal{A}\left[Z^{\perp}\right]$ by $\mathbb{E}_Z = e_Z|_{\mathcal{A}[V]}$.

For $T \in \mathcal{A}[Z]^{\gamma}$ we want $\mathbb{E}_Z(T) = T$, equivalently $e_Z(T1) = T1$ (by faithfulness of 1),

$$i.e. \quad \mathbb{E}_Z(T)1 = e_Z(T1) = T1 \Rightarrow \mathbb{E}_Z(T) = T$$

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Again, using the faithfulness of 1, we can show that indeed $\mathbb{E}_Z(T) = T$ and $\mathbb{E}_Z$ maps onto $\mathcal{A}\left[Z^{\perp}\right]$.

Proof

With this identification, we can perform our work in the Clifford Hilbert space.

Using a basis argument, and analyzing convergence within the Hilbert space, we arrive at the desired identity.

Again, using the faithfulness of 1, we can show that indeed $\mathbb{E}_Z(T) = T$ and $\mathbb{E}_Z$ maps onto $\mathcal{A}\left[Z^{\perp}\right]$.

Hence, $\mathcal{A}\left[Z\right]^{\gamma} = \mathcal{A}\left[Z^{\perp}\right]$

Thank you!

References

[1] P. L. Robinson, *Twisted Duality for Clifford Algebras*, arXiv:1407.1420v1 (2014)

[2] A. M. Sinclair and R. R. Smith, *Finite von Neumann Algebras and Masas*, Cambridge University Press, (2008)