Quotients of multipliers in complete Pick spaces

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SEAM 2017

The classical Smirnov class

Let

$$H^\infty = \{\varphi : \mathbb{D} \to \mathbb{C} : \varphi \text{ is analytic and bounded}\}.$$

Every $\varphi \in H^\infty$ induces a multiplication operator on the Hardy space

$$H^2 = \Big\{ f = \sum_{n=0}^\infty a_n z^n \in \mathcal{O}(\mathbb{D}) : ||f||^2 = \sum_{n=0}^\infty |a_n|^2 < \infty \Big\}.$$

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Fact

N^+ contains H^2.

The Smirnov class of an RKHS

Let $\mathcal{H}$ be a reproducing kernel Hilbert space on a set X with reproducing kernel K, i.e.

$$f(x) = \langle f, K(\cdot, x) \rangle_{\mathcal{H}} \quad (f \in \mathcal{H}, x \in X).$$

Assume K normalized at $x_0 \in X$ (i.e. $K(x, x_0) = 1$ for all $x \in X$).

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Definition

$$N^+(\mathcal{H}) = \left\{ \frac{\varphi}{\psi} : \varphi, \psi \in \mathrm{Mult}(\mathcal{H}), \overline{\psi \mathcal{H}} = \mathcal{H} \right\}.$$

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Is $\mathcal{H} \subset N^+(\mathcal{H})$?

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Example

Let $\mathcal{H} = L^2_a = \mathcal{O}(\mathbb{D}) \cap L^2(\mathbb{D})$ be the Bergman space on $\mathbb{D}$.
Then $\mathrm{Mult}(L^2_a) = H^\infty$ and $L^2_a \not\subset N^+(L^2_a)$.

Nevanlinna-Pick interpolation

Theorem (Pick 1916, Nevanlinna 1919)

Let $z_1, \dots, z_n \in \mathbb{D}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. There exists $f \in H^\infty$ with

$$f(z_i) = \lambda_i \text{ for } 1 \le i \le n \quad \text{and} \quad ||f||_\infty \le 1$$

if and only if the matrix

$$\Big[\frac{1 - \lambda_i \overline{\lambda_j}}{1 - z_i \overline{z_j}} \Big]_{i,j=1}^n$$

is positive.

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$$\left[\frac{1 - \lambda_i \overline{\lambda_j}}{1 - z_i \overline{z_j}} \right]_{i,j=1}^n = \left[(1 - \lambda_i \overline{\lambda_j}) K(z_i, z_j) \right]_{i,j=1}^n$$

is positive.

Here $K(z,w) = (1 - z\overline{w})^{-1}$ is the reproducing kernel of H^2.

Complete Pick spaces

Let $\mathcal{H}$ be a reproducing kernel Hilbert space on a set X with kernel K. Given $z_1, \ldots, z_n \in X$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, does there exist $f \in \text{Mult}(\mathcal{H})$ with

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$$f(z_i) = \lambda_i \quad \text{for } 1 \leq i \leq n \quad \text{and} \quad ||f||_{\text{Mult}(\mathcal{H})} \leq 1?$$

A necessary condition is that the matrix

$$\left[K(z_i, z_j)(1 - \lambda_i \overline{\lambda_j})\right]_{i,j=1}^{n}$$

is positive.

Definition

$\mathcal{H}$ is called a Pick space if this condition is sufficient. $\mathcal{H}$ is called a complete Pick space if the analogue of this condition for matrix valued functions is sufficient.

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with norm $||f||^2_{\mathcal{D}} = ||f'||^2_{L^2(\mathbb{D})} + ||f||^2_{H^2}$. This is a complete Pick space.

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- The Drury-Arveson space H^2_d is the reproducing kernel Hilbert space on $\mathbb{B}_d$, the open unit ball in $\mathbb{C}^d$, with kernel

$$K(z,w) = \frac{1}{1 - \langle z, w \rangle}.$$

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Functions as quotients of multipliers

Recall that

$$N^+(\mathcal{H}) = \left\{ \frac{\varphi}{\psi} : \varphi, \psi \in \mathrm{Mult}(\mathcal{H}), \overline{\psi \mathcal{H}} = \mathcal{H} \right\}.$$

Theorem (Aleman–H.–McCarthy–Richter)

Let $\mathcal{H}$ be a complete Pick space on X whose kernel is normalized at $x_0 \in X$. Then $\mathcal{H} \subset N^+(\mathcal{H})$.

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Let $\mathcal{H}$ be a complete Pick space on X whose kernel is normalized at $x_0 \in X$. Then $\mathcal{H} \subset N^+(\mathcal{H})$.

For the Drury-Arveson space H^2_d with $d < \infty$, this was shown by Alpay, Bolotnikov and Kaptanoğlu.

Corollary

Every function in the Dirichlet space is a quotient of two multipliers of the Dirichlet space.

Zero sets

Let S be a set of functions on X. A subset $Z \subset X$ is called a zero set for S if there exists $f \in S$ with $Z = f^{-1}(0)$.

Corollary

Let $\mathcal{H}$ be a complete Pick space on X whose kernel is normalized at $x_0 \in X$. Then the zero sets for $\mathcal{H}$ and for $\text{Mult}(\mathcal{H})$ agree.

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Proof.

Since $\mathrm{Mult}(\mathcal{H}) \subset \mathcal{H}$, every zero set for $\mathrm{Mult}(\mathcal{H})$ is a zero set for $\mathcal{H}$. Conversely, let $f \in \mathcal{H}$. By the theorem, write $f = \frac{\varphi}{\psi}$ with $\varphi, \psi \in \mathrm{Mult}(\mathcal{H})$ and ψ non-vanishing. Then $f^{-1}(0) = \varphi^{-1}(0)$. □

The corona theorem

Let $\mathcal{H}$ be a reproducing kernel Hilbert space on X. Let

$$\mathcal{M}(\mathrm{Mult}(\mathcal{H})) = \{\rho \in \mathrm{Mult}(\mathcal{H})^* \setminus \{0\} : \rho \text{ is multiplicative}\}.$$

Identify $X \hookrightarrow \mathcal{M}(\mathrm{Mult}(\mathcal{H}))$ via point evaluations.

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The unit disc $\mathbb{D}$ is weak-$*$ dense in $\mathcal{M}(\mathrm{Mult}(\mathcal{D}))$.

Theorem (Costea–Sawyer–Wick, 2011)

For $d < \infty$, the unit ball $\mathbb{B}_d$ is weak-$*$ dense in $\mathcal{M}(\mathrm{Mult}(H^2_d))$.

Spaces on the closed disc

A rotationally invariant space on $\overline{\mathbb{D}}$ is an RKHS $\mathcal{H}$ on $\overline{\mathbb{D}}$ with kernel of the form

$$K(z, w) = \sum_{n=0}^{\infty} a_n (z\overline{w})^n,$$

where

- $a_0 = 1$ and $a_1 > 0$,
- $\sum_{n=0}^{\infty} a_n < \infty$, and
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Then $\operatorname{Mult}(\mathcal{H}) \subset C(\overline{\mathbb{D}})$, so $\overline{\mathbb{D}}$ is a compact subset of $\mathcal{M}(\operatorname{Mult}(\mathcal{H}))$.

Corona question

Is $\mathcal{M}(\operatorname{Mult}(\mathcal{H})) = \overline{\mathbb{D}}$?

The Salas space

Theorem (Aleman–H.–McCarthy–Richter)

There exists a rotationally invariant complete Pick space $\mathcal{H}$ on $\overline{\mathbb{D}}$ such that the corona theorem fails for $\mathcal{H}$, i.e. $\overline{\mathbb{D}} \subsetneq \mathcal{M}(\mathrm{Mult}(\mathcal{H}))$.

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In fact, there exists $\varphi \in \mathrm{Mult}(\mathcal{H})$ which is bounded below, but $\frac{1}{\varphi} \notin \mathrm{Mult}(\mathcal{H})$.

The proof uses a construction of Hector Salas.

Spaces on the closed disc

Proposition

Let $\mathcal{H}$ be a rotationally invariant complete Pick space on $\overline{\mathbb{D}}$. TFAE

(i) $\mathcal{H} = \mathrm{Mult}(\mathcal{H})$ as vector spaces.

(ii) The corona theorem holds for $\mathrm{Mult}(\mathcal{H})$, i.e. $\mathcal{M}(\mathrm{Mult}(\mathcal{H})) = \overline{\mathbb{D}}$.

(iii) The one-function corona theorem holds for $\mathrm{Mult}(\mathcal{H})$, i.e. if $\varphi \in \mathrm{Mult}(\mathcal{H})$ is bounded below on $\overline{\mathbb{D}}$, then $\frac{1}{\varphi} \in \mathrm{Mult}(\mathcal{H})$.

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(iii) $\Rightarrow$(i) Let $f \in \mathcal{H}$, write $f = \frac{\varphi}{\psi}$ with $\varphi, \psi \in \mathrm{Mult}(\mathcal{H})$ and ψ non-vanishing. Since ψ is continuous on $\overline{\mathbb{D}}$, ψ is bounded below, so $\frac{1}{\psi} \in \mathrm{Mult}(\mathcal{H})$ by assumption. Thus $f = \varphi \cdot \frac{1}{\psi} \in \mathrm{Mult}(\mathcal{H})$. □

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A question

Thank you!

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Theorem (Agler–McCarthy, 2000)

Every normalized complete Pick space can be identified with $H^2_\infty\big|_X$ for some set $X \subset \mathbb{B}_\infty$.

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Questions

- Does the corona theorem hold for H^2_∞?
- Does the one-function corona theorem hold for H^2_∞?

Thank you!