Norm of the Backward Shift and Related Operators in Hardy and Bergman Spaces

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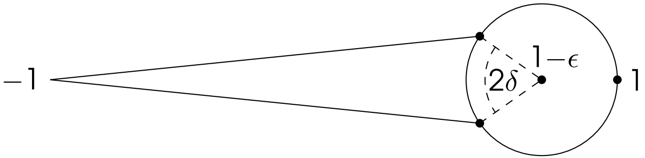
Suppose f is a function defined on the unit disc.

Define $\mathcal{B}(f) = f - f(0)$.

Define $B(f) = [f - f(0)]/z$.

The norm of each of these operators can be at most 2 on any Hardy space H^p (or real harmonic Hardy space h^p). But is it actually this large?

The H^∞ case

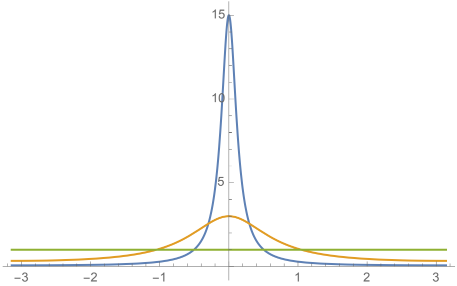


Consider the conformal mapping f of the unit disc onto the region shown, that maps 0 to $1-\epsilon$ and 1 to 1 if extended to the boundary. Then $\|f\|_\infty = 1$ and $\|f\|_\infty = 2-\epsilon$.

Thus $\|B\|_{H^\infty} = 2$.

The h^1 case

For harmonic functions in h^1, $\mathcal{B}(f)$ will be large when most of the "mass" of f is concentrated in a small set on which f is large and has nearly constant sign.



The H^1 case

For h^1 (the space of harmonic functions with bounded M_1 integral means), the Poisson kernel P has $\|\mathcal{B}P\|_{h^1} = 2$. However, the analytic completion of P is not in H^1, so perhaps $\|\mathcal{B}\|_{H^1} < 2$.

In fact, this is true. It is related to the fact that H^1 functions that have mass concentrated on a small set must have signs that oscillate a lot on this set.

First H^1 concentration theorem and a Bound

Theorem

Suppose that $\|f\|_{H^1} = 1$ *and that for some set* $A \subset \mathbb{T}$*, we have* $\int_A Re\, f\, dt/2\pi \geq 1 - \epsilon$ *for* $\epsilon < 1/4$*. Then*

$$m(A) \geq \max_{0<\gamma<1} \frac{\log(\gamma + \epsilon) - \log(1 - 2\epsilon)}{\log \gamma}$$

Theorem

The norm of $\mathcal{B}$ *on the Hardy space* H^1 *is at most* 1.952396.

Second H^1 concentration theorem and a Bound

Theorem

Suppose that $f \in H^1$ and that $\|f\|_{H^1} = 1$. Furthermore, suppose that $\|f\|_{L^1(E)} \geq 1 - \epsilon$ for some set $E \subset \mathbb{T}$ where $m(E) \leq \delta$ and $0 < \epsilon, \delta < 1/2$. Then

$$|f(0)| \leq \left(\frac{1-\epsilon}{\delta}\right)^{\delta} \left(\frac{\epsilon}{1-\delta}\right)^{1-\delta}.$$

Theorem

The norm of the backward shift operator on H^1 is at most 1.7047.

In the bound above, the first term is at most 1.4 if $\epsilon = \delta$, the second is small for small ϵ and δ. Also, the bound decreases as ϵ and δ decrease.

Bergman spaces

Theorem

Suppose that the norm of the operator $\mathcal{B}$ is equal to K on H^p. Let μ be a radial weight such that $\mu(\mathbb{D}) < \infty$. Then the norm of $\mathcal{B}$ is at most K on the Bergman space $A^p(\mu)$.

Theorem

Suppose that the norm of the operator B is equal to K on H^1. Let μ be a finite radial weight that is increasing. Then the norm of B is at most $2K$ on the Bergman space $A^1(\mu)$.

Lemma

Let μ be an increasing radial measure. Then $\|zf\|_{A^1(\mu)} \geq (1/2)\|f\|_{A^1(\mu)}$.

The real harmonic Bergman space

Theorem

The norm of the backward shift on the real harmonic Bergman space $a^1_{\mathbb{R}}$ is at most 1.835. In fact, the same estimate holds on any subspace X of L^1 with the property that $u \in X$ implies that $|u(re^{i\theta})| \leq (1-r)^{-2}$ and the property that the average value of any $u \in X$ on circles centered at the origin is constant.

To prove the theorem, we basically use the fact that the mass of u on the circle of radius r cannot be too concentrated on a set of small angular measure, due to the bound on the size of u.

The operator $\mathcal{B}_r$.

We define the operator $\mathcal{B}_r : h^1_{\mathbb{R}} \to h^1_{\mathbb{R}}$ to be the operator $f \mapsto (\mathcal{B}f)_r$, where $(\mathcal{B}f)_r(z) = (\mathcal{B}f)(rz)$. Equivalently $\mathcal{B}_r$ can be thought of as the operator obtained by applying $\mathcal{B}$ and then restricting the function obtained to the circle centered at the origin with radius r.

Suppose that f and g are functions defined on the interval $[a, b]$. By the convolution of f and g, we mean the function

$$f * g(x) = \frac{1}{b-a} \int_a^b \widetilde{f}(y)\widetilde{g}(x-y)\, dy,$$

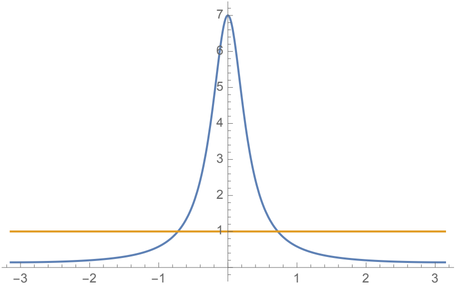
where $\widetilde{f}$ and $\widetilde{g}$ are the periodic extensions of f and g to the real line.

A useful lemma

Lemma

Let f be a real function with average μ and let ν be a finite measure. Then

$$\int |f-\mu|\, d\nu = 2\int_{\{x:f>\mu\}} f-\mu\, d\nu.$$



Theorem

Suppose we are given an $m \times n$ matrix A. Let μ be a fixed number, and let $C_j = \sum_{k=1}^{m} a_{kj}$. Define $D_j = \max(C_j - \mu, 0)$, and let $D = \sum_{j=1}^{n} D_j$. Suppose that $D_j \geq D_k$ but $a_{ij} \leq a_{ik}$. Then we do not decrease D by interchanging a_{ij} and a_{ik}.

The continuous form of the above theorem yields the following results:

Lemma

Suppose that P and f are nonnegative integrable functions on $[a, b]$ and that $\|f\|_1 = 1$, where $\|\cdot\|_1$ denotes the L^1 norm with normalized Lebesgue measure. Then
$\|P * f - P * f * 1\|_1 \leq \|P - P * 1\|_1$.

Corollary

Suppose that u is a nonnegative harmonic function in the unit disc and that $0 \le r < 1$*. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} |u(re^{i\theta}) - u(0)|\, d\theta \le u(0)\frac{1}{2\pi}\int_0^{2\pi} |P_r(e^{i\theta}) - 1|\, d\theta = u(0)\left(2 - \frac{4}{\pi}\arccos(r)\right).$$

where P_r *is a Poisson kernel.*

Note that this corollary holds for $r = 1$ if we replace $\frac{1}{2\pi}\int_0^{2\pi} |u(e^{i\theta}) - u(0)|\, d\theta$ by $\|u - u(0)\|_{h^1_{\mathbb{R}}}$.

We must now deal with functions that are allowed to be negative. The following is a continuous form of a previous theorem.

Lemma

Suppose that P is a nonnegative integrable function on $[\alpha, \beta]$ and f is an integrable function such that $\|f\|_1 = 1$, where $\|\cdot\|_1$ denotes the L^1 norm with normalized Lebesgue measure. Let P^* denote the decreasing rearrangement of P, so that $P^*(t) = \inf\{x : m(\{y \in [\alpha, \beta] : P(y) > x\}) \leq t\}$. Then $\|P * f - P * f * 1\|_1 \leq \|Q - Q * 1\|_1$ where Q is some function of the form $aP^*(x) - bP^*(\alpha + \beta - x)$, where $a + b = 1$.

We need to have a condition under which we can conclude that $Q(x) = P^*(x)$. The following theorem provides such a condition.

Theorem

Suppose that P is a continuous function on $[\alpha, \beta]$ and that P is nonnegative with average 1 and decreasing.
Let $Q(x) = aP(x) - bP(\beta + \alpha - x)$ for some nonnegative real numbers a and b such that $a + b = 1$.
Then there is a c between α and β such that $aP(c) - bP(\beta + \alpha - c) = a - b$.
If for some such c,

$$\int_{\alpha}^{\alpha+c} P\,dx + \int_{\beta-c}^{\beta} P\,dx \geq 2c,$$

then $\|P - P * 1\|_1 \geq \|Q - Q * 1\|_1$, where $\|\cdot\|_1$ denotes the L^1 norm with normalized Lebesgue measure.

Theorem

Suppose that $0 \le r < 1$ *and* $u \in h^1_{\mathbb{R}}$. *Then*

$$\frac{1}{2\pi}\int_0^{2\pi} |u(re^{i\theta}) - u(0)|\, d\theta \le \inf_{a\in\mathbb{R}} \|u - a\|_{h^1_{\mathbb{R}}} \frac{1}{2\pi}\int_0^{2\pi} |P_r(e^{i\theta}) - 1|\, d\theta$$
$$= \inf_{a\in\mathbb{R}} \|u - a\|_{h^1_{\mathbb{R}}} \left(2 - \frac{4}{\pi}\arccos(r)\right)$$

where P_r *is a Poisson kernel.*

Note that the theorem holds for $r = 1$ if we replace $\frac{1}{2\pi}\int_0^{2\pi} |u(e^{i\theta}) - u(0)|\, d\theta$ by $\|u - u(0)\|_{h^1_{\mathbb{R}}}$.

Corollary

The value of $\|\mathcal{B}\|_{h^1_{\mathbb{R}} \to a^1_{\mathbb{R}}} = 1$.