Convex-Cyclicity & Convex-Polynomial Approximation

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33^{rd} South Eastern Analysis Meeting
University of Tennessee
March 17-19, 2017

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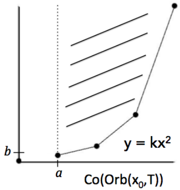

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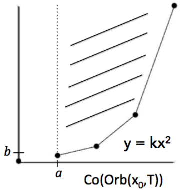

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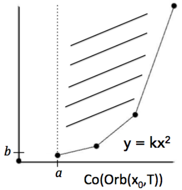

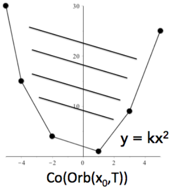

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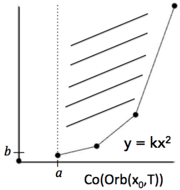

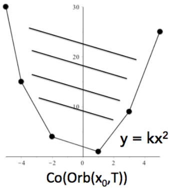

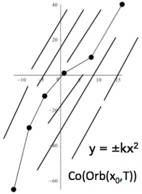

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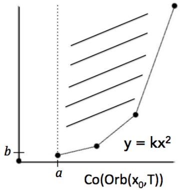

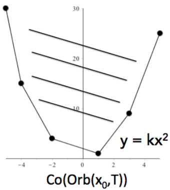

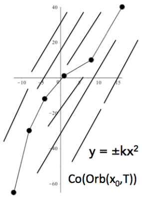

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T is convex-cyclic $\Leftrightarrow \exists\ x_0$ s.t. $co\{T^n x_0\}_{n=0}^{\infty}$ is dense.

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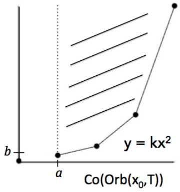
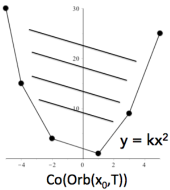
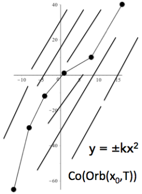
Convex-Cyclic

Every Invariant Convex Set is a subspace!







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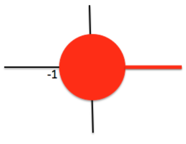
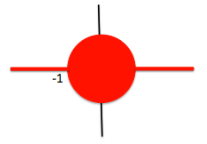
Theorem (Matrices on $\mathbb{R}^n$; Elsner '82, Feldman-McGuire '15)

If T is a matrix acting on $\mathbb{R}^n$, then every invariant convex set for T is an invariant subspace for T if and only if $\sigma(T) \subseteq \mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R}^+)$.



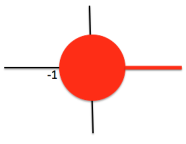
Theorem (Matrices on $\mathbb{C}^n$; Feldman-McGuire '15)

If T is a matrix acting on $\mathbb{C}^n$ with eigenvalues $\{\lambda_j\}_{j=1}^n$, then every invariant convex set for T is an invariant subspace for T if and only if $\sigma(T) \subseteq \mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R})$ and $\lambda_j \neq \overline{\lambda}_k$ for all $1 \leq j, k \leq n$.



Theorem (Convex-Cyclic Matrices on $\mathbb{R}^n$)

A matrix T acting on $\mathbb{R}^n$ is convex-cyclic if and only if T is cyclic and $\sigma(T) \subseteq \mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R}^+)$. Furthermore, the convex-cyclic vectors for T are the same as the cyclic vectors for T.



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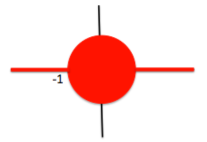

Convex-Cyclicity & Convex-Polynomials

- cyclicity $\leftrightarrow$ invariant subspaces

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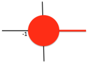

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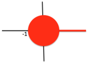

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The following hold for $1 \leq p < \infty$

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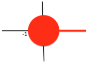

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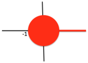

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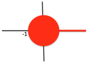

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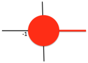

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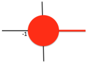

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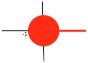

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then M_z is convex-cyclic on $L^2(\mu)$ and has a dense G_δ set of convex-cyclic vectors.

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Theorem (Discrete Measures on $\mathbb{C}$)

If $\mu = \sum_{k=1}^\infty w_k \delta_{a_k}$, $w_k > 0$, $\sum_k w_k < \infty$, and

1. $\{a_k\}_{k=1}^\infty \subseteq \mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R})$
2. $a_j \neq \overline{a_k}$ *for all* j, k,

then M_z is convex-cyclic on $L^2(\mu)$ and has a dense G_δ set of convex-cyclic vectors. So, $\exists f \in L^2(\mu)$ s.t. $cl\{pf : p \in \mathcal{CP}\} = L^2(\mu)$ and thus

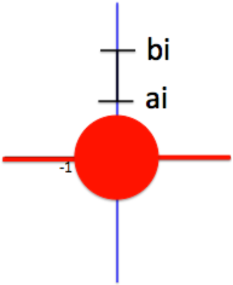
$$\mathcal{CP}^2(|f|^2 d\mu) = P^2(|f|^2 \mu) = L^2(\mu).$$

Complex Convex-Polynomial Approximation

Theorem (Complex Convex Stone-Weierstrass Theorems)

The following hold for $1 \le p < \infty$ and $0 < a < b$

1. $\mathcal{CP}$ *is weak***-dense in* $L^\infty(ai, bi)$ *if and only if* $a \ge 1$.
2. $\mathcal{CP}$ *is dense in* $L^p(ai, bi)$ *if and only if* $a \ge 1$.
3. $\mathcal{CP}$ *is dense in* $C[ai, bi]$ *if and only if* $a > 1$.
4. $\mathcal{CP}$ *is dense in* $L^p(\mu)$, $\mu \in M^+(i, bi)$ *if and only if* $\mu(\{i\}) = 0$.



Thanks for your time!

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