Nonlinear Interaction of Trapped Shallow Water Waves over Idealized Bottom Boundaries

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Abstract:

We some consider interesting problems of shallow water waves trapped over a varying bottom boundary. Using the Hamiltonian formulation for free surface waves of an irrotational, incompressible fluid, we investigate the dispersion relations of the linear solution for some examples and their associated resonance manifolds in the search for resonant triad clusters.

Formulation of the problem

Let $\Phi(r,z,t)$ represent the velocity potential on the domain $D \subset \mathbb{R}^3$ bounded above by the free surface $\eta(r,t)$ and below by $h(r)$ and over the positive half plane. The velocity potential and the free surface satisfy the governing equations for irrotational invisid flow.

Continuity of the velocity potential:

$$\Delta\Phi = 0. \tag{0.1}$$

The bottom boundary condition $(z = h(r))$:

$$\frac{\partial \Phi}{\partial z} + \nabla h \cdot \nabla \Phi = 0 \quad z = -h \tag{0.2}$$

The surface boundary conditions $(z = \eta(r,t))$:

$$\begin{aligned} \frac{\partial \Psi}{\partial t} &= -g\eta - \frac{1}{2}|\nabla\Psi|^2 \quad z = \eta \\ \frac{\partial \eta}{\partial t} &= -\nabla\eta \cdot \nabla\Psi + \frac{\partial \Psi}{\partial z} \end{aligned}$$

Trapped wave boundary conditions:

$$\begin{aligned} \varphi &< \infty \quad \text{at } r = r_0 \\ \varphi &\to 0 \quad \text{as } r \to \infty. \end{aligned}$$

Hamiltonian formulation

The Hamiltonian for continuous media:

$$\mathscr{H} = \int_D H d\mathbf{r} \quad H = T + V. \tag{0.3}$$

Kinetic and potential energy:

$$T = \frac{1}{2} \int_{-h}^{\eta} dz \left[|\nabla \Phi|^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right], V = \frac{1}{2} g \eta^2 \tag{0.4}$$

Zakharov [1968] showed that η and $\psi = \Phi|_{z=\eta}$ are canonical variables for this Hamiltonian and the canonical equations can be written as

$$\frac{\partial \eta}{\partial t} = \frac{\delta \mathscr{H}}{\delta \Psi}, \quad \frac{\partial \Psi}{\partial t} = -\frac{\delta \mathscr{H}}{\delta \eta}. \tag{0.5}$$

Shallow Water

One difficulty of the general problem comes from the variable integration boundaries. However, our domain of interest is near the coast in water of shallow depth which means that the velocity potential will not vary with the depth.

The Hamiltonian for shallow water becomes

$$\mathscr{H} = \frac{1}{2} g \int_{\mathbb{R}^+} \eta^2 + \frac{1}{2} (\eta + h) |\nabla \Phi|^2. \tag{0.6}$$

The canonical equations yield the shallow water dynamic and kinematic boundary conditions:

$$\frac{\partial \eta}{\partial t} = -\nabla \cdot [(\eta + h) \nabla \psi] \quad \frac{\partial \psi}{\partial t} = -g\eta - \frac{1}{2} (\nabla \Psi)^2$$

Fourier transform the canonical variables along the $y-$axis.

$$\Psi(x,y,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x,t) e^{iky} dk \quad \eta(x,y,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \zeta(x,t) e^{iky} dk \tag{0.7}$$

Then the Hamiltonian becomes:

$$\begin{aligned}
\mathscr{H} &= \frac{1}{2} g \int_D \int_{-\infty}^{\infty} \zeta_1 \zeta_1^* dk dx \\
&+ \frac{1}{2} \int_D \int_{-\infty}^{\infty} \left(\frac{\partial \psi_1}{\partial x} \frac{\partial \psi_1^*}{\partial x} + k_n^2 \psi_1 \psi_1^* \right) h dk_{12} dx \\
&+ \frac{1}{2\sqrt{2\pi}} \int_D \int_{-\infty}^{\infty} \left(\zeta_3 \frac{\partial \psi_1}{\partial x} \frac{\partial \psi_2}{\partial x} - k_1 k_2 \zeta_3 \psi_1 \psi_2 \right) \delta(k_1 + k_2 + k_3) dk_{123} dx.
\end{aligned}$$

Linear system

The canonical equations yield the linear shallow water equations

$$\begin{aligned} \zeta_t + (h\psi_x)_x - k^2 h\psi &= 0 \\ \psi_t + g\zeta &= 0. \end{aligned}$$

We can combine these to eliminate ζ and form the so called Sturm Liouville edgewave equation where we write $c^2 = gh$ as the local phase velocity and $\psi_{tt} = -\omega^2\psi$

$$\frac{\partial}{\partial x}\left(c^2 \frac{\partial \psi_n}{\partial x}\right) + \left(\omega_n^2 - c^2 k_n^2\right)\psi_n = 0. \tag{0.8}$$

Trapped wave boundary conditions

$$\begin{aligned} \psi < \infty \quad & \text{at } r = 0 \\ \psi \to 0 \quad & \text{as } r \to \infty. \end{aligned}$$

Sturm-Liouville Fourier expansion.

$$\psi = \sum_n A_n F_n e^{i(k_n y - \omega_n t)} \tag{0.9}$$

Examples

Piecewise shelf profile

The depth profile for this example is a discontinuous step shelf in the x direction with the discontinuity at the position x_s.

$$h(x) = \begin{cases} h_1 & x \leq x_s \\ h_2 & x > x_s \end{cases} \tag{0.10}$$

The general solution satisfying the boundary conditions that describes trapped waves on the shelf is

$$\psi = \sum_{n \in \mathbb{Z}} e^{i(k_n y - \omega_n t)} \begin{cases} A_n \left(e^{i\gamma_1 x} + e^{-i\gamma_1 x}\right) & \gamma_1^2 < 0 \\ B_n e^{-\gamma_2 x} & \gamma_2^2 \geq 0 \end{cases} \tag{0.11}$$

where $\gamma_j = \sqrt{\omega^2 - g h_j k^2}$.
Matching the solutions and the mass flux at the discontinuity x_s with $\alpha = \frac{h_2 \gamma_2}{h_1 \gamma_1}$
We can write this as

$$\begin{bmatrix} \left(e^{i\gamma_1 x} + e^{-i\gamma_1 x}\right) & -e^{-\gamma_2 x} \\ \left(e^{i\gamma_1 x} - e^{-i\gamma_1 x}\right) & \alpha e^{-\gamma_2 x} \end{bmatrix} \begin{bmatrix} A_n \\ B_n \end{bmatrix} = 0. \tag{0.12}$$

$$\alpha \left(e^{i\gamma_1 x} + e^{-i\gamma_1 x}\right) e^{-\gamma_2 x} + \left(e^{i\gamma_1 x} - e^{-i\gamma_1 x}\right) e^{-\gamma_2 x} = 0$$
$$\tan\left(\gamma_1 x_s\right) + \frac{h_2 \gamma_2}{h_1 \gamma_1} = 0$$

$$\Omega\left(\omega_n, k_n\right) = \tan\left(\gamma_1 x_s\right) + \frac{h_2 \gamma_2}{h_1 \gamma_1} = 0 \tag{0.13}$$

This has a discrete set of real solutions (ω_n, k_n) that yield the dispersion relation for the trapped waves over a shelf.

Constant slope profile

The example for edgewaves over a depth profile of constant slope, h(x)=sx was worked out by Ursell in 1952.

The general solution is a linear combination of Laguerre polynomials.

$$\Phi = \sum_{n\in\mathbb{Z}} A_n e^{i(k_n y-\omega_n t)} e^{-k_n x} L\left(2k_n x\right) \tag{0.14}$$

The dispersion relation for a constant slope profile takes the approximate form

$$\omega_j^2 = (2j+1)\, gsk, \quad j = 0, 1, 2, \ldots \tag{0.15}$$

We note that this relation has the nice form

$$\omega_j\left(k\right) = a_j \left|\kappa\right|^{1/p} \quad p\in\mathbb{N} \tag{0.16}$$

In polar coordinates with a circularly symmetric depth profile $h = h(r)$

Top hat island

In polar coordinates with a circularly symmetric depth profile $h = h(r)$. The depth profile for this example is a discontinuous step in the radial direction with circular contours and a shelf break at the position r_s.

$$h(r) = \begin{cases} h_1 & r \leq r_s \\ h_2 & r > r_s \end{cases} \tag{0.17}$$

The general solution is a linear combination of Bessel functions with integer mode wavenumbers.

$$\Phi = \sum e^{i(n\theta - \omega_n t)} \begin{cases} A_n J_n(k_1 r) & 0 \leq r < r_s \quad \text{on the shelf} \\ B_n H_n^1(k_2 r) & r > r_s \quad \text{off the shelf.} \end{cases} \tag{0.18}$$

The matching condition at the discontinuity yields the dispersion relation

$$\Omega(\omega_n, n) = J_n(\mu\omega_n) H_n'(\varepsilon\mu\omega_n) - \varepsilon J_n'(\mu\omega_n) H_n(\varepsilon\mu\omega_n) = 0. \tag{0.19}$$

It can be shown using properties of Bessel functions that this yields only complex eigenfrequencies. Numerical investigations due to Longuet-Higgins (1967) have shown that there is a class of dispersion curves with an imaginary component just below that real axis. Therefore these eigenfrequencies describe waves with a slow dissipation.

Nonlinear interaction

Using the linear eigenfunctions, the Hamiltonian becomes

$$\mathscr{H} = \sum_n \left(\alpha_n A_n^2 + \beta_n B_n^2\right) + \sum_{lnm} \gamma_{lnm} A_l B_n B_m \delta\,(l+n+m) \tag{0.20}$$

Transform the amplitudes to complex conjugate variables

$$A_n^p = \frac{1}{\sqrt{2}} \left(\frac{\omega_n^p}{g}\right)^{\frac{1}{2}} (a_n^p + a_n^{p*}) \quad B_n^p = \frac{i}{\sqrt{2}} \left(\frac{\omega_n^p}{g}\right)^{-\frac{1}{2}} (a_n^p - a_n^{p*})$$

Bogoliubov transformation brings $\mathscr{H}_2$ to the standard quadratic form and $\mathscr{H}_3$ to a cubic form.

$$\mathscr{H}_2 = \sum_{pn} \omega_n^p \,|F_n^p|^2 \,\|a_n^p\|^2 \quad \mathscr{H}_3 = \sum_{spq} \sum_{lnm} \Gamma_{lnm}^{spq} \,(a_l^s a_n^p a_n^q - a_l^s a_n^p a_m^{q*} - a_l^s a_n^{p*} a_m^q + a_l^s a_n^{p*} a_m^{q*}) + c.c. \tag{0.21}$$

Interaction coefficient

$$\Gamma_{lnm}^{spq} = \frac{g}{32\sqrt{2}\pi^3} \left(\frac{\omega_l^s}{\omega_n^p \omega_m^q}\right)^{\frac{1}{2}} \int_R \left(F_l \frac{\partial F_n}{\partial r} \frac{\partial F_m}{\partial r} + \frac{nm}{r^2} F_l F_n F_m\right) r dr. \tag{0.22}$$

The cubic Hamilton

Thus the evolution equations for the cubic Hamiltonian can now be written as a single evolution equation

$$i\frac{\partial a_l^{s*}}{\partial t}=\sum_{pn}\sum_{qm}\Gamma_{lnm}^{spq}\left(a_n^p a_m^q+a_n^p a_m^{q*}\right)+c.c.$$

where the modes satisfy the so called resonance conditions:

$$\begin{aligned} k_n+k_m\pm k_l&=0\\ \omega\left(k_n\right)+\omega\left(k_m\right)\pm\omega\left(k_l\right)&=0. \end{aligned}$$

Resonant triads

The interaction equations for a single triad are an integrable system describing the evolution of energy between modes.

$$\frac{\partial a_l^{1*}}{\partial t} = -i\Gamma_{lnm}^{123} a_n^2 a_m^3$$
$$\frac{\partial a_l^{2*}}{\partial t} = -i\Gamma_{lnm}^{213} a_n^1 a_m^3$$
$$\frac{\partial a_l^{3*}}{\partial t} = -i\Gamma_{lnm}^{312} a_n^1 a_m^2$$

The stability of the system depends on the initial conditions and the interaction coefficients. However the interaction coefficients hinge on the dispersion relation.

The resonance conditions for a single triad yield the so called resonance manifold

$$G_{s,p,q}(\kappa_1,\kappa_2) = \Omega_s(|k_1|) + \Omega_p(|k_2|) - \Omega_q(|k_1+k_2|) = 0. \tag{0.23}$$

The resonance manifold for the constant slope example yields a very nice relationship: $k_2 = \beta_{spq} k_1$. Hence resonant triads for this problem have the form:

$$\{k_1, \beta k_1, (1+\beta) k_1\} \tag{0.24}$$

This yields a class of resonant triad clusters for each given wavenumber k.

Furthermore, the resonance manifold of any dispersion relationship of the form $\omega_j(k) = a_j |\kappa|^{1/p} \quad p \in \mathbb{N}$ yields a similar condition for the triad modes.

Thank you

References

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