Bounded point derivations on $R^p(X)$

Stephen Deterding
Joint work with James Brennan

University of Kentucky

Table of contents

1. Uniform rational approximation
2. L^p rational approximation
3. Proof sketch

Uniform rational approximation

Runge's Theorem

- Throughout this talk X will always be a compact subset in the complex plane.

Runge's Theorem

- Throughout this talk X will always be a compact subset in the complex plane.

Runge's Theorem (1885)

Let A be a set of points which includes at least one point from every unbounded connected component of $\mathbb{C} \setminus X$. If f is a function which is analytic in a neighborhood of X, then there exists a sequence of rational functions which converges to f uniformly. Moreover, if those rational functions have poles, the poles are points of A.

Runge's Theorem

- Throughout this talk X will always be a compact subset in the complex plane.

Runge's Theorem (1885)

Let A be a set of points which includes at least one point from every unbounded connected component of $\mathbb{C} \setminus X$. If f is a function which is analytic in a neighborhood of X, then there exists a sequence of rational functions which converges to f uniformly. Moreover, if those rational functions have poles, the poles are points of A.

Question

If the hypotheses of Runge's theorem are strengthened will rational approximation hold for a larger set of functions such as continuous functions on X?

The function space $R(X)$

- $R(X)$ is the uniform closure of the rational functions with poles off X.

The function space $R(X)$

- $R(X)$ is the uniform closure of the rational functions with poles off X.
- $C(X)$ is the space of continuous functions on X.

The function space $R(X)$

- $R(X)$ is the uniform closure of the rational functions with poles off X.
- $C(X)$ is the space of continuous functions on X.
- The problem of uniform rational approximation is to find conditions on X so that $R(X) = C(X)$.

The function space $R(X)$

- $R(X)$ is the uniform closure of the rational functions with poles off X.

- $C(X)$ is the space of continuous functions on X.

- The problem of uniform rational approximation is to find conditions on X so that $R(X) = C(X)$.

- If X contains an interior point, then X contains an open set. Every function in $R(X)$ is analytic on this open set and hence $R(X) \neq C(X)$. Thus from now on we can assume that X contains no interior.

Bounded point derivations on $R(X)$

- A bounded point derivation D on $R(X)$ at a point x_0 is a bounded linear functional on $R(X)$ such that $D(fg) = D(f)g(x_0) + D(g)f(x_0)$ for all functions f, g belonging to $R(X)$.

Bounded point derivations on $R(X)$

- A bounded point derivation D on $R(X)$ at a point x_0 is a bounded linear functional on $R(X)$ such that $D(fg) = D(f)g(x_0) + D(g)f(x_0)$ for all functions f, g belonging to $R(X)$.

- The existence or nonexistence of bounded point derivations on $R(X)$ shows how much differentiability is preserved under uniform convergence.

Bounded point derivations and the usual derivative

- A natural question to ask is: what is the relationship between a bounded point derivation and the usual notion of a derivative?

Bounded point derivations and the usual derivative

- A natural question to ask is: what is the relationship between a bounded point derivation and the usual notion of a derivative?

- This question was first considered by Wang, who proved that if there is a bounded point derivation on $R(X)$ at x_0 and f belongs to $R(X)$ then f has an approximate derivative at x_0.

Approximate derivatives

- A function f has an approximate derivative at x_0 if there exists a subset E with *full area density* at x_0, and a number L such that

$$\lim_{x \to x_0, x \in E} \left| \frac{f(x) - f(x_0)}{x - x_0} - L \right| = 0$$

Approximate derivatives

- A function f has an approximate derivative at x_0 if there exists a subset E with *full area density* at x_0, and a number L such that

$$\lim_{x \to x_0, x \in E} \left| \frac{f(x) - f(x_0)}{x - x_0} - L \right| = 0$$

- We say that L is the approximate derivative at x_0.

Approximate derivatives

- A function f has an approximate derivative at x_0 if there exists a subset E with *full area density* at x_0, and a number L such that

$$\lim_{x \to x_0, x \in E} \left| \frac{f(x) - f(x_0)}{x - x_0} - L \right| = 0$$

- We say that L is the approximate derivative at x_0.
- Let $\Delta_n(x_0)$ be the ball $\{x : |x - x_0| < \frac{1}{n}\}$. A set E is said to have full area density at x_0 if

$$\lim_{n \to \infty} \frac{m(\Delta_n(x_0) \setminus E)}{m(\Delta_n(x_0))} = 0$$

Wang's theorem

Wang's theorem

- **Theorem (Wang 1973)**

  Suppose there is a bounded point derivation on $R(X)$ at x_0. Let D denote this bounded point derivation. Then for $f \in R(X)$ there exists a set E of full area density at x_0 such that

$$\lim_{x \to x_0, x \in E} \left| \frac{f(x) - f(x_0)}{x - x_0} - Df \right| = 0$$

Wang's theorem

- **Theorem (Wang 1973)**

  Suppose there is a bounded point derivation on $R(X)$ at x_0. Let D denote this bounded point derivation. Then for $f \in R(X)$ there exists a set E of full area density at x_0 such that

$$\lim_{x \to x_0, x \in E} \left| \frac{f(x) - f(x_0)}{x - x_0} - Df \right| = 0$$

  - Wang's theorem is false if the approximate derivative is replaced with the usual derivative.

Wang's theorem

- **Theorem (Wang 1973)**

  Suppose there is a bounded point derivation on $R(X)$ at x_0. Let D denote this bounded point derivation. Then for $f \in R(X)$ there exists a set E of full area density at x_0 such that

  $$\lim_{x \to x_0, x \in E} \left| \frac{f(x) - f(x_0)}{x - x_0} - Df \right| = 0$$

  - Wang's theorem is false if the approximate derivative is replaced with the usual derivative.
  - The reason for this is it is a result of Dolzhenko that there exists a nowhere differentiable function in $R(X)$ whenever X is a nowhere dense set.

L^p rational approximation

The space $R^p(X)$

- We now consider approximation in the L^p norm where $1 \leq p < \infty$

The space $R^p(X)$

- We now consider approximation in the L^p norm where $1 \leq p < \infty$
- For $1 \leq p < \infty$, $R^p(X)$ is the closure of the rational functions with poles off X in the L^p norm.

The space $R^p(X)$

- We now consider approximation in the L^p norm where $1 \leq p < \infty$
- For $1 \leq p < \infty$, $R^p(X)$ is the closure of the rational functions with poles off X in the L^p norm.
- The problem of L^p rational approximation is to find conditions on X so that $R^p(X) = L^p(X)$.

Bounded point evaluations on $R^p(X)$

- A point is admits a bounded point evaluation if the map from f to $f(x_0)$ can be extended from the space of rational functions with poles off X to $R^p(X)$.

Bounded point evaluations on $R^p(X)$

- A point is admits a bounded point evaluation if the map from f to $f(x_0)$ can be extended from the space of rational functions with poles off X to $R^p(X)$.

- The following theorem demonstrates the importance of bounded point evaluations in the study of rational approximation in the L^p norm.

Bounded point evaluations on $R^p(X)$

- A point is admits a bounded point evaluation if the map from f to $f(x_0)$ can be extended from the space of rational functions with poles off X to $R^p(X)$.

- The following theorem demonstrates the importance of bounded point evaluations in the study of rational approximation in the L^p norm.

Theorem

For $p \neq 2$, $R^p(X) = L^p(X)$ if and only if there are no bounded point evaluations on X.

Bounded point derivations on $R^p(X)$

- A bounded point derivation on $R^p(X)$ is defined slightly differently than a bounded point derivation on $R(x)$.

Bounded point derivations on $R^p(X)$

- A bounded point derivation on $R^p(X)$ is defined slightly differently than a bounded point derivation on $R(x)$.
- One can't define a bounded point derivation on $R^p(X)$ by the formula $D(fg) = D(f)g(x_0) + D(g)f(x_0)$ since $R^p(X)$ is not closed under multiplication.

Bounded point derivations on $R^p(X)$

- A bounded point derivation on $R^p(X)$ is defined slightly differently than a bounded point derivation on $R(x)$.
- One can't define a bounded point derivation on $R^p(X)$ by the formula $D(fg) = D(f)g(x_0) + D(g)f(x_0)$ since $R^p(X)$ is not closed under multiplication.
- Instead there is a bounded point derivation on $R^p(X)$ if the map from f to $f'(x_0)$ can be extended from the space of rational functions with poles off X to $R^p(X)$.

Bounded point derivations on $R^p(X)$

- A bounded point derivation on $R^p(X)$ is defined slightly differently than a bounded point derivation on $R(x)$.
- One can't define a bounded point derivation on $R^p(X)$ by the formula $D(fg) = D(f)g(x_0) + D(g)f(x_0)$ since $R^p(X)$ is not closed under multiplication.
- Instead there is a bounded point derivation on $R^p(X)$ if the map from f to $f'(x_0)$ can be extended from the space of rational functions with poles off X to $R^p(X)$.
- It can be shown that if x_0 admits a bounded point derivation on $R^p(X)$ then x_0 admits a bounded point evaluation on $R^p(X)$.

Bounded point derivations on $R^p(X)$

- A bounded point derivation on $R^p(X)$ is defined slightly differently than a bounded point derivation on $R(x)$.
- One can't define a bounded point derivation on $R^p(X)$ by the formula $D(fg) = D(f)g(x_0) + D(g)f(x_0)$ since $R^p(X)$ is not closed under multiplication.
- Instead there is a bounded point derivation on $R^p(X)$ if the map from f to $f'(x_0)$ can be extended from the space of rational functions with poles off X to $R^p(X)$.
- It can be shown that if x_0 admits a bounded point derivation on $R^p(X)$ then x_0 admits a bounded point evaluation on $R^p(X)$.
- The existence or nonexistence of bounded point derivations on $R^p(X)$ shows how much differentiability is preserved under convergence in the L^p norm.

Main Theorem

- Our goal is to extend Wang's theorem from $R(X)$ to $R^p(X)$

Main Theorem

- Our goal is to extend Wang's theorem from $R(X)$ to $R^p(X)$

Theorem

For $2 < p < \infty$, suppose that there is a bounded point derivation on $R^p(X)$ at x_0. Let D denote this bounded point derivation. Then for $f \in R(X)$ there exists a set E of full area density at x_0 such that

$$\lim_{x \to x_0, x \in E} \left| \frac{f(x) - f(x_0)}{x - x_0} - Df \right| = 0$$

Main Theorem

- Our goal is to extend Wang's theorem from $R(X)$ to $R^p(X)$

Theorem

For $2 < p < \infty$, suppose that there is a bounded point derivation on $R^p(X)$ at x_0. Let D denote this bounded point derivation. Then for $f \in R(X)$ there exists a set E of full area density at x_0 such that

$$\lim_{x \to x_0, x \in E} \left| \frac{f(x) - f(x_0)}{x - x_0} - Df \right| = 0$$

- Note that this theorem only applies when $2 < p < \infty$

What about $p \leq 2$

- If $p < 2$ then $R^p(X) = L^p(X)$.

What about $p \leq 2$

- If $p < 2$ then $R^p(X) = L^p(X)$.

- This implies that no point of X can admit a bounded point derivation.

What about $p \leq 2$

- If $p < 2$ then $R^p(X) = L^p(X)$.

- This implies that no point of X can admit a bounded point derivation.

- It is unknown whether the theorem is true for the case of $p = 2$.

Proof sketch

The set of full area density

- In order to show that there is an approximate derivative at x_0, we must construct a set with full area density at x_0 to take the limit over.

The set of full area density

- In order to show that there is an approximate derivative at x_0, we must construct a set with full area density at x_0 to take the limit over.

- To do this, we first list the inequalities that we need in our proof and then show that the set of points where these inequalities is satisfied has full area density at x_0. This is the set E of full area density.

Starting the proof

- Let $L_x(f) = \dfrac{f(x) - f(x_0)}{x - x_0} - Df$

Starting the proof

- Let $L_x(f) = \dfrac{f(x) - f(x_0)}{x - x_0} - Df$
- To prove the main theorem we need to show that $L_x(f)$ tends to 0 as x tends to x_0 through E.

Starting the proof

- Let $L_x(f) = \dfrac{f(x) - f(x_0)}{x - x_0} - Df$
- To prove the main theorem we need to show that $L_x(f)$ tends to 0 as x tends to x_0 through E.
- If f is a rational function with poles off X then $Df = f'(x_0)$ and $L_x(f)$ converges to 0 as x converges to x_0 through E.

Starting the proof

- Let $L_x(f) = \dfrac{f(x) - f(x_0)}{x - x_0} - Df$
- To prove the main theorem we need to show that $L_x(f)$ tends to 0 as x tends to x_0 through E.
- If f is a rational function with poles off X then $Df = f'(x_0)$ and $L_x(f)$ converges to 0 as x converges to x_0 through E.
- Let $\{f_j\}$ be a sequence of rational functions which converges to f in the L^p norm. Then $|L_x(f)| \leq |L_x(f - f_j)| + |L_x(f_j)|$.

Starting the proof

- Let $L_x(f) = \dfrac{f(x) - f(x_0)}{x - x_0} - Df$
- To prove the main theorem we need to show that $L_x(f)$ tends to 0 as x tends to x_0 through E.
- If f is a rational function with poles off X then $Df = f'(x_0)$ and $L_x(f)$ converges to 0 as x converges to x_0 through E.
- Let $\{f_j\}$ be a sequence of rational functions which converges to f in the L^p norm. Then $|L_x(f)| \leq |L_x(f - f_j)| + |L_x(f_j)|$.
- If we can show that $|L_x(f - f_j)| \leq C||f - f_j||_p$ then we can show that $L_x(f)$ tends to 0 as x tends to x_0 through E.

Representing measures

- Key to our proof is the concept of a representing measure

Representing measures

- Key to our proof is the concept of a representing measure

Theorem

If ℓ is a linear functional on $R^p(X)$ then there exists a function k in $L^q(X)$ such that

$$\int_X f(z) k dA = \ell(f)$$

Representing measures

- Key to our proof is the concept of a representing measure

Theorem

If ℓ is a linear functional on $R^p(X)$ then there exists a function k in $L^q(X)$ such that

$$\int_X f(z) k dA = \ell(f)$$

- The measure kdA is called a representing measure for the linear functional.

Representing measures

- Key to our proof is the concept of a representing measure

Theorem

If ℓ is a linear functional on $R^p(X)$ then there exists a function k in $L^q(X)$ such that

$$\int_X f(z)k\,dA = \ell(f)$$

- The measure kdA is called a representing measure for the linear functional.
- If the linear functional is a bounded point evaluation on $R^p(X)$ at x_0, then we say that kdA is a representing measure for x_0

Representing measures

- Key to our proof is the concept of a representing measure

Theorem

If ℓ is a linear functional on $R^p(X)$ then there exists a function k in $L^q(X)$ such that

$$\int_X f(z)k\,dA = \ell(f)$$

- The measure kdA is called a representing measure for the linear functional.
- If the linear functional is a bounded point evaluation on $R^p(X)$ at x_0, then we say that kdA is a representing measure for x_0
- If k_1dA is a representing measure for a bounded point derivation on $R^p(X)$ at x_0 and $k = (z - x_0)k_1$, then kdA is a representing measure for x_0.

Representing measures for points in E.

- To show that $|L_x(f - f_j)| \leq C||f - f_j||_p$, it is enough to get a bound on the difference quotient since the bounded point derivation is already a bounded linear functional.

Representing measures for points in E.

- To show that $|L_x(f - f_j)| \leq C||f - f_j||_p$, it is enough to get a bound on the difference quotient since the bounded point derivation is already a bounded linear functional.
- To bound the difference quotient, we construct a representing measure for x in E. Our construction is based off the following theorem.

Representing measures for points in E.

- To show that $|L_x(f - f_j)| \leq C||f - f_j||_p$, it is enough to get a bound on the difference quotient since the bounded point derivation is already a bounded linear functional.
- To bound the difference quotient, we construct a representing measure for x in E. Our construction is based off the following theorem.

Theorem (Bishop 1959)

Let μ be an annihilating measure for $R(X)$ and suppose that $\hat{\mu}(x) = \int_X \frac{d\mu(z)}{z - x}$ is defined and nonzero. Then $\nu = \frac{1}{\hat{\mu}(x)} \frac{\mu}{z - x}$ is a representing measure for x.

Representing measures for points in E.

- To show that $|L_x(f - f_j)| \leq C||f - f_j||_p$, it is enough to get a bound on the difference quotient since the bounded point derivation is already a bounded linear functional.
- To bound the difference quotient, we construct a representing measure for x in E. Our construction is based off the following theorem.

Theorem (Bishop 1959)

Let μ be an annihilating measure for $R(X)$ and suppose that $\hat{\mu}(x) = \int_X \frac{d\mu(z)}{z - x}$ is defined and nonzero. Then $\nu = \frac{1}{\hat{\mu}(x)} \frac{\mu}{z - x}$ is a representing measure for x.

- In our case, k is a representing measure for x_0 so $(z - x_0)k$ is an annihilating measure and our representing measure for x is slightly different from the one in Bishop's theorem.

Bounding the difference quotient

- We focus on getting the appropriate bound for $|f(x) - f(x_0)|$.

Bounding the difference quotient

- We focus on getting the appropriate bound for $|f(x) - f(x_0)|$.
- We can use the representing measures that we have for x and x_0 to get the desired bound.

Bounding the difference quotient

- We focus on getting the appropriate bound for $|f(x) - f(x_0)|$.
- We can use the representing measures that we have for x and x_0 to get the desired bound.
- These measures have the nice property that as x gets close to x_0, the representing measures for x get close to the representing measure for x_0 in the following sense.

Bounding the difference quotient

- We focus on getting the appropriate bound for $|f(x) - f(x_0)|$.
- We can use the representing measures that we have for x and x_0 to get the desired bound.
- These measures have the nice property that as x gets close to x_0, the representing measures for x get close to the representing measure for x_0 in the following sense.
- If k_x is the representing measure for x then $\left|\int fk_x dA - \int fk dA\right| \leq C|x - x_0| \cdot ||f||_p$

Bounding the difference quotient

- We focus on getting the appropriate bound for $|f(x) - f(x_0)|$.
- We can use the representing measures that we have for x and x_0 to get the desired bound.
- These measures have the nice property that as x gets close to x_0, the representing measures for x get close to the representing measure for x_0 in the following sense.
- If k_x is the representing measure for x then $\left|\int fk_x dA - \int fk dA\right| \leq C|x - x_0| \cdot ||f||_p$

Theorem

If f belongs to $R^p(X)$ and x is in E then $|f(x) - f(x_0)| \leq C|x - x_0| \cdot ||f||_p$

Conclusion

- If there is a bounded point derivation on $R^p(X)$ at x_0, it is not true that every function in $R^p(X)$ is differentiable at x_0.

Conclusion

- If there is a bounded point derivation on $R^p(X)$ at x_0, it is not true that every function in $R^p(X)$ is differentiable at x_0.

- For $2 < p < \infty$, if there is a bounded point derivation on $R^p(X)$ at x_0, then every function in $R^p(X)$ has an *approximate* derivative at x_0.

Conclusion

- If there is a bounded point derivation on $R^p(X)$ at x_0, it is not true that every function in $R^p(X)$ is differentiable at x_0.

- For $2 < p < \infty$, if there is a bounded point derivation on $R^p(X)$ at x_0, then every function in $R^p(X)$ has an *approximate* derivative at x_0.

- For $1 \leq p < 2$, there are no bounded point derivations on $R^p(X)$.

Conclusion

- If there is a bounded point derivation on $R^p(X)$ at x_0, it is not true that every function in $R^p(X)$ is differentiable at x_0.
- For $2 < p < \infty$, if there is a bounded point derivation on $R^p(X)$ at x_0, then every function in $R^p(X)$ has an *approximate* derivative at x_0.
- For $1 \leq p < 2$, there are no bounded point derivations on $R^p(X)$.
- It is not known whether the existence of a bounded point derivation on $R^2(X)$ at x_0 implies that every function in $R^2(X)$ has an approximate derivative at x_0.

References

- Bishop, E. *A Minimal Boundary for Function Algebras*. Pacific J. Math. 9 (1959), 629 - 642
- Browder, Andrew *Point Derivations on Function Algebras*. J. Functional Analysis 1, (1967) 22-27
- Dolzhenko, E.P. *Construction on a nowhere dense continuum of a nowhere differentiable function which can be expanded into a series of rational functions*.(Russian) Dokl. Akad. Nauk SSSR 125 1959 970-973.
- Wang, James Li Ming *An Approximate Taylor's Theorem for R(X)*. Math. Scand. 33 (1973), 343-358