Choquet order and hyperrigidity for function systems

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joint work with Matthew Kennedy

$1 \in \mathcal{S} = \mathcal{S}^* \subset \mathrm{C}(X)$ is a function system.
$K = \{\varphi \in \mathcal{S}^* : \varphi \geq 0,\ \varphi(1) = 1\}$ state space, compact, convex,
and $x \in X \longrightarrow \varepsilon_x \in K$, where $\varepsilon_x(f) = f(x)$ for $f \in \mathcal{S}$.

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Theorem (Kadison 1951)

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$\partial S := \partial K$ extreme points is Choquet boundary of $\mathcal{S}$.
$f \in \mathcal{S}$ affine on K, so $\mathcal{S} \longrightarrow \mathrm{C}(\overline{\partial K})$ completely isometric.
$\overline{\partial K}$ is the Shilov boundary of $\mathcal{S}$.

By Hahn-Banach and Riesz Representation Theorems,
for $\varphi \in K$ there exists $\mu \in M_+(\overline{\partial K})$ representing measure:

$$\varphi(f) = \int f \, d\mu \qquad \text{for } f \in \mathcal{S}.$$

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Definition

Choquet order: $\mu \prec_c \nu$ in $M_+(K)$ if $\int f\, d\mu \le \int f\, d\nu$ for f convex.

This implies that $\int f\, d\mu = \int f\, d\nu$ for $f \in \mathcal{S}$, so represent same φ.

Theorem (Choquet, Mokobodski)

K metrizable.
$\mu \in M_+(K)$ is maximal in $\prec_c \iff \operatorname{supp} \mu \subset \partial K$.

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Mokobodski: this does not characterize maximality.
However, if ∂K is closed, then μ is maximal $\iff \operatorname{supp} \mu \subset \partial K$.

Classical result:

Theorem (Korovkin)

If $\Phi_n : \mathrm{C}[a,b] \to \mathrm{C}[a,b]$ positive maps s.t.

$$\lim_{n\to\infty} \Phi_n(f) = f \quad \text{for} \quad f \in \{1, x, x^2\},$$

then $\lim_{n\to\infty} \Phi_n(f) = f$ for all $f \in \mathrm{C}[a,b]$.

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modern, significant improvement:

Theorem (Arveson)

If $\pi : \mathrm{C}[a,b] \to \mathcal{B}(\mathcal{H})$ $*$-repn., $\Phi_n : \mathrm{C}[a,b] \to \mathcal{B}(\mathcal{H})$ (completely) positive maps s.t.

$$\lim_{n\to\infty} \Phi_n(f) = \pi(f) \quad \textit{for} \quad f \in \{1, x, x^2\},$$

then $\lim_{n\to\infty} \Phi_n(f) = \pi(f)$ *for all $f \in \mathrm{C}[a,b]$.*

Definition

$1 \in F \subset \mathrm{C}(X)$ is a Korovkin set if $\Phi_n : \mathrm{C}(X) \to \mathrm{C}(X)$ are positive,
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F is a strong Korovkin set if $\pi : \mathrm{C}[a,b] \to \mathcal{B}(\mathcal{H})$ $*$-repn.,
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Theorem (Šaškin)

X compact metric. $1 \in F \subset \mathrm{C}(X)$. $\mathcal{S} = \overline{\operatorname{span}}\{F \cup F^*\}$.
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Question (Arveson)

Characterize strong Korovkin sets.

Definition

$1 \in \mathcal{S} = \mathcal{S}^* \subset \mathfrak{A} = \mathrm{C}^*(\mathcal{S})$ is hyperrigid if whenever
$\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ $*$-repn, and $\Phi_n : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ c.p.
$\lim_{n\to\infty} \Phi_n(s) = \pi(s)$ for $s \in \mathcal{S} \implies \lim_{n\to\infty} \Phi_n(a) = \pi(a)$ for $a \in \mathfrak{A}$.

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$\pi|_{\mathcal{S}}$ has unique extension property (u.e.p.)
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Theorem (Arveson)

$1 \in \mathcal{S} = \mathcal{S}^* \subset \mathfrak{A} = \mathrm{C}^*(\mathcal{S})$. Then
$\mathcal{S}$ is hyperrigid $\iff$ $\pi|_{\mathcal{S}}$ has u.e.p. $\forall \pi$ $*$-repn.

Definition

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Conjecture (Arveson)

$\mathcal{S}$ is hyperrigid $\iff$ every irreducible $*$-repn. is a boundary repn.

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Remark

For $1 \in \mathcal{S} = \mathcal{S}^* \subset \mathrm{C}(X)$, this asks if $\partial\mathcal{S} = X$,
is $\mathcal{S}$ is a strong Korovkin set in $\mathrm{C}(\partial\mathcal{S})$?

Definition

Dilation order: $\mu \prec_d \nu \in M_+(K)$ if there exist $*$-repns.

$\pi : \mathrm{C}(K) \to \mathcal{B}(\mathcal{H}), \quad \xi \in \mathcal{H}, \quad \langle \pi(f)\xi, \xi\rangle = \int f\, d\mu \quad \forall f \in \mathrm{C}(K)$

$\sigma : \mathrm{C}(K) \to \mathcal{B}(\mathcal{K}), \quad \eta \in \mathcal{K}, \quad \langle \sigma(f)\eta, \eta\rangle = \int f\, d\nu \quad \forall f \in \mathrm{C}(K)$

and isometry $J : \mathcal{H} \to \mathcal{K}$ s.t. $J\xi = \eta$ and $J^*\sigma(f)J = \pi(f)\ \forall f \in \mathcal{S}$.

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Corollary

$\mu \prec_c \nu \iff \exists \Phi : \mathrm{C}(K) \to L^\infty(\mu)$ positive s.t.

1. $\Phi(f) = f$ for all $f \in A(K)$, and
2. $\int \Phi(f) \, d\mu = \int f \, d\nu$ for all $f \in \mathrm{C}(K)$.

$\pi_\mu : \mathrm{C}(K) \to \mathcal{B}(L^2(\mu))$ by $\pi(f) = M_f$.

Theorem 2

π_μ has u.e.p. $\iff$ μ is maximal in $\prec_d$.

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Corollary (Hyperrigidity for function systems)

If ∂S is closed, then $\mathcal{S}$ is hyperrigid in $\mathrm{C}(\partial S)$.

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Corollary

If X is metrizable, $1 = \mathcal{S} \subset \mathrm{C}(X)$, $\pi : \mathrm{C}(X) \to \mathcal{B}(\mathcal{H})$ $*$-repn.
Then π has u.e.p. $\iff$ π is supported on ∂S.

Application to approximation theory

The following does not require metrizability, so it generalizes Šaškin's Theorem even in the classical situation.

Corollary

$1 \in \mathcal{S} = \overline{\text{span}}\{F \cup F^*\} \subset \mathrm{C}(X)$.

TFAE

1. $\partial S = X$
2. F is a Korovkin set.
3. F is a strong Korovkin set.

Application to classical Choquet theory

Theorem (Cartier)

If K is metrizable, $\mu \prec_c \nu$, then $\exists \lambda : K \to M_{+,1}(K)$ s.t.

1. $x \to \lambda_x(f)$ *is Borel* $\forall f \in \mathrm{C}(K)$,
2. $\lambda_x(f) = f(x) \quad \forall f \in A(K)$, *and*
3. $\int f \, d\nu = \int \lambda_x(f) \, d\mu \quad \forall f \in \mathrm{C}(K)$.

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Theorem 3

K compact convex, $\mu \prec_c \nu$, then $\exists \lambda : K \to M_{+,1}(K)$ s.t.

1. $x \to \lambda_x(f)$ is Borel $\forall f \in \mathrm{C}(K)$,
2. $\lambda_x(f) = f(x)$ a.e.$(\mu) \quad \forall f \in A(K)$, and
3. $\int f \, d\nu = \int \lambda_x(f) \, d\mu \quad \forall f \in \mathrm{C}(K)$.

Thank you.

The end.