Purity of absolutely continuous constrained commuting row contractions

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Joint work with Ken Davidson.

Properties of the functional calculus

Let $A(\mathbb{D})$ be the disc algebra, which consists of holomorphic functions on the unit disc $\mathbb{D} \subset \mathbb{C}$ which are continuous on $\overline{\mathbb{D}}$. For $f \in A(\mathbb{D})$ we put

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QUESTION When does Φ_T extend to be weak-$*$ continuous on $H^\infty(\mathbb{D})$?

The answer: $A(\mathbb{D})$–Henkin measures

We can decompose

$$\mathcal{H} = \mathcal{H}_{cnu} \oplus \mathcal{H}_u$$
$$T = T_{cnu} \oplus U$$

where U is unitary and T_{cnu} has no invariant subspace on which it restricts to be unitary (it is completely non-unitary).

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Theorem (Sz.-Nagy–Foias)

There is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}_{cnu}$ and a unitary operator $W \in B(\mathcal{K})$ with the property that

$$f(T_{cnu}) = P_{\mathcal{H}_{cnu}} f(W)|_{\mathcal{H}_{cnu}}, \quad f \in \mathbb{C}[z].$$

Moreover, the spectral measure of W is absolutely continuous with respect to Lebesgue measure on the circle.

A direct calculation shows that the functional calculus for W admits a weak-$*$ continuous extension to $H^\infty(\mathbb{D})$, and thus the same holds for T_{cnu}.

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What about U? Let μ denote its spectral measure on the circle $\mathbb{T} \subset \mathbb{C}$. The property we seek is easily checked to be equivalent to the following:

$$\lim_{n\to\infty} \int_{\mathbb{T}} f_n d\mu = 0$$

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Such measures are called $A(\mathbb{D})$–Henkin.

Characterization of $A(\mathbb{D})$–Henkin measures

Hence, the functional calculus of $T = T_{cnu} \oplus U$ extends weak-$*$ continuously to $H^\infty(\mathbb{D})$ if and only if the spectral measure of U is $A(\mathbb{D})$–Henkin.

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QUESTION When is a measure $A(\mathbb{D})$–Henkin?

Let σ denote normalized Lebesgue measure on $\mathbb{T}$. By the Cauchy formula,

$$f(0) = \int_{\mathbb{T}} f d\sigma$$

for every $f \in A(\mathbb{D})$. Clearly, σ is an $A(\mathbb{D})$–Henkin measure.

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Theorem (Henkin 1968, Cole-Range 1972)

A regular Borel measure μ is $A(\mathbb{D})$–Henkin if and only it is absolutely continuous with respect to σ.

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A regular Borel measure μ is $A(\mathbb{D})$–Henkin if and only it is absolutely continuous with respect to σ.

The proof is based on a rather precise description of the dual space $A(\mathbb{D})^*$. In fact, the characterization of Henkin measure works in several variables as well (for the ball algebra $\mathrm{A}(\mathbb{B}_\mathrm{d})$).

The correct multivariate function theoretic framework: the Drury-Arveson space

Let H^2_d denote the Drury-Arveson space. This is the reproducing kernel Hilbert space on the open unit ball $\mathbb{B}_d$ with kernel

$$k(z,w) = \frac{1}{1 - \langle z, w\rangle}, \quad z, w \in \mathbb{B}_d \,.$$

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The algebra $\mathcal{M}_d$ of multipliers consists of the functions $\varphi : \mathbb{B}_d \to \mathbb{C}$ such that $\varphi H^2_d \subset H^2_d$. Each multiplier $\varphi \in \mathcal{M}_d$ defines a multiplication operator on

$$M_\varphi : H^2_d \to H^2_d.$$

In particular, $\mathcal{M}_d \subset B(H^2_d)$ is a weak-$*$ closed operator algebra.

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Denote by $\mathcal{A}_d \subset \mathcal{M}_d$ the closure of the polynomials in the multiplier norm,

Multivariate operator theory

Let $T_1, \dots, T_d \in B(\mathcal{H})$ be commuting operators such that $T = (T_1, \dots, T_d)$ is a contraction, which is equivalent to

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When Φ_T extends weak-$*$ continuously to $\mathcal{M}_d$, we say that T is absolutely continuous.

$\mathcal{A}_d$–Henkin functionals

Definition

We say that a functional $\Psi \in \mathcal{A}_d^*$ is $\mathcal{A}_d$–*Henkin* if $\Psi(f_n) \to 0$ whenever $\{f_n\}_n \subset \mathcal{A}_d$ converges to 0 in the weak-$*$ topology of $\mathcal{M}_d$.

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Special case: Ψ is given as integration against a measure μ on the sphere $\mathbb{S}_d$. Since $\|f\|_\infty \leq \|f\|_{\mathcal{M}_d}$, the condition of Ψ being $\mathcal{A}_d$–Henkin is a priori weaker than μ being an $A(\mathbb{B}_d)$–Henkin. Therefore, the Henkin-Cole-Range characterization does not apply.

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CONJECTURE (C.-DAVIDSON 2016) A measure μ on $\mathbb{S}_d$ is $\mathrm{A}(\mathbb{B}_\mathrm{d})$–Henkin if and only if it induces an $\mathcal{A}_d$–Henkin functional.

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Theorem (Hartz 2017)

There is a measure μ on $\mathbb{S}_d$ which is not $\mathrm{A}(\mathbb{B}_\mathrm{d})$–Henkin yet it induces an $\mathcal{A}_d$–Henkin functional.

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Theorem (C.-Davidson 2016)

There is a commutative von Neumann algebra $\mathfrak{W}$ such that

$$\mathcal{A}_d^{**} \cong \mathcal{M}_d \oplus \mathfrak{W}, \quad \mathcal{A}_d^* \cong (\mathcal{M}_d)_* \oplus_1 \mathfrak{W}_*.$$

The other extreme: the space $\mathfrak{W}_*$

What are $\mathfrak{W}_*$ and $\mathfrak{W}$?

Theorem (C.-Davidson 2016)

Let $\Psi \in \mathfrak{W}_*$. Then there exists a measure μ which is singular with respect to every positive representing measure for the origin, with $\|\mu\| = \|\Psi\|$ and such that

$$\Psi(f) = \int_{\mathbb{S}_d} f d\mu, \quad f \in \mathcal{A}_d \,.$$

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Moreover, the is a spherical unitary $u = (u_1, \dots, u_d)$ such that $\mathfrak{W} = W^*(u_1, \dots, u_d)$.

A measure μ for which the associated functional $\Psi \in \mathcal{A}_d^*$ belongs to $\mathfrak{W}_*$ will be called $\mathcal{A}_d$–totally singular.

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Corollary (C.-Davidson 2016)

Let $T = (T_1, \dots, T_d)$ be a commuting row contraction. Then, we can write $T = A \oplus S$ where $A = (A_1, \dots, A_d)$ is absolutely continuous and $S = (S_1, \dots, S_d)$ is a spherical unitary with an $\mathcal{A}_d$–totally singular spectral measure.

In particular, a completely non-unitary commuting row contraction is absolutely continuous.

Coextensions and purity

Theorem (Sz.-Nagy, Wold-von Neumann)

Let $T \in B(\mathcal{H})$ be a contraction. Then, there is a cardinal number γ and a unitary operator U such that

$$T^* = (S^{*(\gamma)} \oplus U^*)|_{\mathcal{H}}$$

where S denotes the unilateral (Hardy) shift.

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The contraction T is pure (or of class $C_{\bullet 0}$) if U is absent. Equivalently, T is pure if $\lim_{N\to\infty} \|T^{*N}\xi\| = 0$ for every $\xi \in \mathcal{H}$.

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where $M = (M_1, \ldots, M_d)$ denotes the Drury-Arveson shift.

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Constrained contractions

Let $T = (T_1, \ldots, T_d)$ be a commuting row contraction on $\mathcal{H}$. Let

$$\Phi_T : \mathcal{A}_d \to B(\mathcal{H})$$

be the associated functional calculus. Let $\mathcal{J} \subset \mathcal{A}_d$ be a non-trivial closed ideal. Then, T is said $\mathcal{J}$-constrained if $\mathcal{J} = \ker \Phi_T$.

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Theorem (Folklore)

Let $T \in B(\mathcal{H})$ be a $\mathcal{J}$-constrained absolutely continuous contraction. Then, T is pure.

This fact is very important in the development of the functional model theory of constrained contractions.

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Example

Let U denote the bilateral shift and let $T = (U, 0)$. Then, T is an absolutely continuous commuting row contraction, which is clearly not pure since U is unitary. Moreover, if we denote by $\mathcal{J} = \overline{z_2 \mathcal{A}_2}$, then we see that T is $\mathcal{J}$-constrained.

Purity of constrained row contractions

Given a non-trivial closed ideal $\mathcal{J} \subset \mathcal{A}_d$, we denote by $Z(\mathcal{J})$ its zero set on the closed ball.

Theorem (C.-Davidson 2016)

Let $T = (T_1, \dots, T_d)$ be an absolutely continuous $\mathcal{J}$-constrained commuting row contraction. Then, there is a cardinal γ and some spherical unitary $U = (U_1, \dots, U_d)$ with an $\mathcal{A}_d$–Henkin spectral measure concentrated on $Z(\mathcal{J}) \cap \mathbb{S}_d$ such that

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We can now identify a condition under which the analogue of the one-variable result holds: the zero set of $\mathcal{J}$ on the sphere must be small relative to $\mathcal{A}_d$.

Corollary (C.-Davidson 2016)

Let $T = (T_1, \ldots, T_d)$ be an absolutely continuous $\mathcal{J}$-constrained commuting row contraction. Assume that $Z(\mathcal{J}) \cap \mathbb{S}_d$ is null with respect to every $\mathcal{A}_d$–Henkin measure. Then, T is pure.

Ideals with small zero sets

Theorem (C.-Davidson 2016)

Let $K \subset \mathbb{S}_d$ be a closed subset which is null with respect to every $\mathcal{A}_d$–Henkin measure. Let $\Lambda \subset \mathbb{B}_d$ be an interpolating sequence for $\mathcal{M}_d$ such that $\overline{\Lambda} \cap \mathbb{S}_d \subset K$. Then, there exists a closed ideal $\mathcal{J} \subset \mathcal{A}_d$ such that $\Lambda \cup K = Z(\mathcal{J})$.

Hence, for such an ideal $\mathcal{J}$, any absolutely continuous $\mathcal{J}$-constrained commuting row contraction is pure (note that $\mathcal{J}$ is not weak-$*$ dense in $\mathcal{M}_d$).

Thank you!