Optimal Polynomial Approximants in Dirichlet Spaces

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This talk is based on joint work with Dmitry Khavinson, Conni Liaw, Daniel Seco, and Brian Simanek.

Outline

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Definition

For $-\infty < \alpha < \infty$, the space D_α consists of all analytic functions $f : \mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\} \to \mathbb{C}$ whose Taylor coefficients in the expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad z \in \mathbb{D},$$

satisfy

$$\|f\|_\alpha^2 = \sum_{k=0}^{\infty} (k+1)^\alpha |a_k|^2 < \infty.$$

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Given two functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ in D_α, we also have the associated inner product

$$\langle f, g \rangle_\alpha = \sum_{k=0}^{\infty} (k+1)^\alpha a_k \overline{b_k}.$$

Examples

- $\alpha = -1$ corresponds to *the Bergman space* A^2, consisting of functions with
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- $\alpha = 1$: *the (classical) Dirichlet space* D of functions whose derivatives have finite area integral:

$$\int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty.$$

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In other words, p_n is an optimal polynomial of order n to $1/f$ if

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where $\mathcal{P}_n$ denotes the space of polynomials of degree at most n.
Note: $p_n f$ is the orthogonal projection of 1 onto the subspace $f \cdot \mathcal{P}_n$. Therefore, optimal approximants p_n always exist and are unique for any nonzero function f, and any degree $n \geq 0$.

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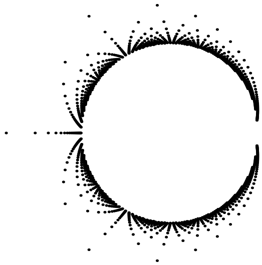
- What can we say about the zeros of the optimal polynomial approximants p_n?
- In particular, what is the smallest modulus of a zero of any optimal polynomial approximant for a given Dirichlet space, and what is the corresponding extremal function?

Here are the zeros of the optimal approximants to $1/(1-z)$ for the Hardy space:



Zeros are always outside a certain disk

Theorem (C.B., Khavinson, Liaw, Seco, Sola, 2016, JLMS)

Let $\alpha \in \mathbb{R}$, let $f \in D_\alpha$ have $f(0) \neq 0$, and let p_n be the optimal approximant to $1/f$ of degree n. Then

- *if $\alpha \geq 0$, all the zeros of the optimal approximants lie outside the closed unit disk;*
- *if $\alpha < 0$, the zeros lie outside the closed disk $\overline{D}(0, 2^{\alpha/2})$.*

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The proof of this theorem relies on the connection between these optimal polynomials and certain weighted reproducing kernels $K_n(z, 0) = p_n(z)f(z)$, where $K_n(z, w)$ is the reproducing kernel for the space $f \cdot \mathcal{P}_n$, but that is not the focus of this talk.

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This theorem begs the question, for $\alpha < 0$, what is the minimum modulus of a zero of an optimal polynomial approximant? Is $2^{\frac{\alpha}{2}}$ sharp?

Minimal zero extremal problem

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$$\inf_{n\in\mathbb{N}} \left\{ |z| : p_n(z) = 0, \|p_n f - 1\|_{A^2} = \min_{q\in\mathcal{P}_n} \|qf - 1\|_{A^2}, \ f \in A^2 \right\}.$$

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- If we rewrite this ratio in terms of the coefficients a_k of f, you can see that it is enough to consider $a_k \geq 0$.

Three term recurrence relationship

By taking partial derivatives of the ratio $\frac{|\langle f, zf\rangle|}{\|zf\|^2_{A^2}}$ with respect to each coefficient a_k (where $f(z) = \sum_{k=0}^{\infty} a_k z^k$) and setting those partial derivatives equal to 0, we see that the coefficients a_k for the extremal function must satisfy a 3 term recurrence relationship, for some real number λ:

$$\begin{cases} a_1 = \lambda a_0 \\ a_{k+1} = \lambda a_k - \frac{k+2}{k+1} a_{k-1} \qquad \text{for } k \geq 1 \end{cases}$$

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$$a_k = P_k(\lambda).$$

Connection with orthogonal polynomials and Jacobi matrices

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Such polynomials P_k are very well-known! Favard's Theorem from the 1920s tells us that such a 3-term recurrence relationship leads to the existence of a compactly supported measure μ on the real line and monic (real) polynomials P_k that are orthogonal with respect to the measure μ

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$$\mathcal{J} := \begin{pmatrix} 0 & \sqrt{3/2} & 0 & 0 & \cdots \\ \sqrt{3/2} & 0 & \sqrt{4/3} & 0 & \cdots \\ 0 & \sqrt{4/3} & 0 & \sqrt{5/4} & \cdots \\ 0 & 0 & \sqrt{5/4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

How do you use this connection to solve the extremal problem?

- Consider the cut-off problem $\mathcal{U}_N := \sup_{f \in \mathcal{P}_N} \frac{|\langle f, zf \rangle|}{\|zf\|_{A^2}^2}$.

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1. *Let* $\alpha \in \mathbb{R}$. *Then*

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2. *If* $\alpha \geq 0$, *then* $\|\mathcal{J}_\alpha\| = 2$ *and there is no solution to the extremal problem.*
3. *If* $\alpha < 0$, *then* $\|\mathcal{J}_\alpha\| > 2$, *and the extremal problem has solution*

$$f(z) = \sum_{n=0}^{\infty} P_n(\|\mathcal{J}_\alpha\|) z^n$$

But how to actually FIND the norm of this Jacobi matrix?

Back to the recurrence relationship that the extremal function satisfies, for $k \geq 1$:

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Multiply both sides by z^k and sum, and get an equation that can be translated into the following simple differential equation:

$$\left(1 - \lambda z + z^2\right) f'(z) + (3z - \lambda) f(z) = 0,$$

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Theorem (BKLSS, 2016)

Let $\alpha = -1$. The minimal modulus of a zero of an optimal approximant is $\frac{2}{3}\sqrt{2}$ and the extremal function $f^*(z)$ is given by $\frac{2\sqrt{2}}{(\sqrt{2}-z)^3}$.

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- We also have results about the *global* behavior of the zeros, where Jentzsch-type theorems hold: namely, given a (say cyclic) function f such that $1/f$ has a singularity on the unit circle, every point on the unit circle is a limit point of the zeros of the optimal polynomial approximants of $1/f$.
- Since optimal polynomial approximants are reproducing kernels for certain weighted spaces, these results about the zeros of the optimal approximants gives insight into the zeros of reproducing kernels.

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- Study the singularities of the functions that are extremal solutions (that is, functions whose optimal polynomial approximants have minimal zeros) in that context.
- Study the geometry of the "paths" along which the zeros of optimal approximants approach the unit circle.
- Examine whether the behavior of the zeros of the optimal approximants of a function can give insight into the cyclicity of the function.

Thank you!