Hardy-Hodge decomposition on manifolds

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$$\mathcal{C}^{\pm}f(\xi) = \pm\frac{f(\xi)}{2} + \lim_{\varepsilon\to 0}\frac{1}{2i\pi}\int_{\Gamma\setminus B(\xi,\varepsilon)}\frac{f(\zeta)}{\zeta-\xi}d\zeta, \qquad a.e.\xi\in\Gamma,$$

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  where $B(\xi,\varepsilon)$ is the ball centered at ξ of radius ε.
- In particular, we have that

$$f(\xi) = \mathcal{C}^{+} - \mathcal{C}^{-}.$$

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- Moreover, if $f \in H^p(D^\pm)$, it can be recovered from its boundary values by the Cauchy formula:

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- Moreover, if $f \in H^p(D^\pm)$, it can be recovered from its boundary values by the Cauchy formula:

$$\mathcal{C}f(z) = \pm\frac{1}{2i\pi}\int_\Gamma \frac{f(\xi)}{\xi - z}d\xi, \qquad z \in D^\pm.$$

- Also, the Cauchy theorem holds:

$$0 = \frac{1}{2i\pi}\int_\Gamma \frac{f(\xi)}{\xi - z}d\xi, \qquad f \in H^p(D^\pm), \qquad z \in D^\mp.$$

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Hence the relation $f(\xi) = \mathcal{C}^+ - \mathcal{C}^-$ yields:

Theorem

For Γ a Lipschitz Jordan curve and $1 < p < \infty$, there holds a topological sum:

$$L^p(\Gamma) = H^p(D^+) \oplus H^p(D^-).$$

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- We begin by reformulating the result in the plane.

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Corollary

For $1 < p < \infty$, a $\mathbb{R}^2$-valued vector field of L^p-class on Γ is uniquely the sum of the trace of the gradient of a harmonic function in D^+ and the trace of the gradient of a harmonic function in D^-, where both gradients have nontangential maximal function in L^p.

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- Elements of $\mathcal{H}^p_{\pm}$ have nontangential limits on $\mathcal{M}$ whose L^p norm is equivalent to the L^p norm of the maximal function [Calderòn 50][Dalberg 77][Jerison-Kenig 82][Verchota 84].

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- One can then consider $\mathcal{D}^p = (\mathcal{G}^q)^\perp$ for the pairing

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Theorem

Let $\mathcal{M}$ be a compact connected Lipschitz hypersurface in $\mathbb{R}^n$ which is locally a graph, and $1 < p < 2 + \varepsilon(\mathcal{M})$. Then, there is a direct sum

$$(L^p(\mathcal{M}))^n = \mathcal{H}^p_+ \oplus \mathcal{H}^p_- \oplus \mathcal{D}^p(\mathcal{M}).$$

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- We call this Hardy-Hodge decomposition for it generalizes both Plemelj decomposition as a sum of Hardy functions (dimension 2) and the Helmholtz-Hodge decomposition for tangent vector fields ($\mathcal{T}^p(\mathcal{M}) = \mathcal{G}^p(\mathcal{M}) \oplus \mathcal{D}^p(\mathcal{M})$) .

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- When $\mathcal{M}$ is smooth the decomposition holds in more general spaces of functions or distributional currents.

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- The Hardy-Hodge decomposition seems new even if $\mathcal{M}$ is smooth.
- We do not know if the restriction $1 < p < 2 + \varepsilon$ in the Lipschitz case is an artefact of our proof.

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- Thus, we are left to decompose $V - D - \nabla u$ which is a normal vector field on $\mathcal{M}$.

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- Let u be harmonic in Ω^+ and solve the Dirichlet problem $u_{|\mathcal{M}} = \psi$ [Verchota,1984]. Here restriction on p appears again. Then $\nabla u \in \mathcal{H}^p_+$ and the tangential component of its nontangential limit on $\mathcal{M}$ is G.
- Thus, we are left to decompose $V - D - \nabla u$ which is a normal vector field on $\mathcal{M}$. For this we use some Clifford analysis and homogeneity arguments.

And most importantly

Thank you!

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- This gives us an analog of the Plemelj formula:

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Theorem

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- Uniqueness follows from uniqueness of the Hodge decomposition and the Liouville theorem for harmonic functions.

Application to the characterization of silent magnetization distributions on a surface

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- What about more general surfaces for which the curvature plays a more decisive role?

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- Of course we could trade e_1 for any other e_j.

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- Writing the Hardy-Hodge decomposition $W = H^+ + H^- + D$, we get by the Cauchy-Clifford formula that harmonic gradients cannot contribute the $e_1 e_2 e_3$ term, and that only D can.

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Then, the potential P_{divm} is zero in the unbounded component of $\mathbb{R}^3 \setminus \mathcal{M}$ if, and only if the left monogenic function

$$H_1^- - \mathcal{C}(D_2 e_2 e_3)$$

has vanishing e_1-component.

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- On Lipschitz surfaces, the result still holds when the range of p is restricted to $p > 2 - \varepsilon$ where ε depends on the Lipschitz constant of $\mathcal{M}$.
- When $\mathcal{M}$ is smooth, substitutes to $L^1(\Gamma)$ and $L^\infty(\Gamma)$ can be taken to be $H^1(\Gamma)$ and $BMO(\Gamma)$.