Free Polynomial Biholomorphisms

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Intro

- Given a tuple of $d \times d$ matrices $A = (A_1, \ldots, A_g)$ and indeterminates $x = (x_1, \ldots, x_g)$ we form

$$L_A(x) = I_d + \sum A_j x_j + \sum A_j^* x_j^*,$$

a **monic** **(affine)** **linear pencil** and

$$\mathcal{D}_A(1) = \{z \in \mathbb{C}^g : L_A(z) \succeq 0\}$$

the associated **spectrahedron** (or **LMI domain**).

- Spectrahedra play a large role in semi-definite programming, convex optimization and real algebraic geometry.

Intro

- We evaluate L on a tuple of $d \times d$ matrices $X = (X_1, \ldots, X_g)$ by

$$L_A(X) = I_d \otimes I_n + \sum A_j \otimes X_j + \sum A_j^* \otimes X_j^*.$$

  $L_A(X)$ is a $dn \times dn$ self-adjoint matrix. For each $n \geq 1$ we set

$$\mathcal{D}_A(n) = \{X \in M_n(\mathbb{C}^g) : L_A(X) \succeq 0\}$$

  and define a **free spectrahedron** (or **free LMI domain**) to be the sequence $(\mathcal{D}_A(n))_{n=1}^{\infty}$.
- Let $\Lambda_A(x) = \sum A_j x_j$, which is the **truly linear pencil**
  - $\Lambda_A(X) = \sum A_j \otimes X_j$

Intro

- Free spectrahedra
  - Arise naturally in systems engineering problems (signal flow diagrams)
  - Are canonical examples of matrix convex sets
  - Are intimately connected to the theory of completely positive maps and operator systems and spaces
- Certain model problems in systems engineering can be described by a system of matrix inequalities
  - For optimization we would like convexity
  - With boundedness we get a free spectrahedron
  - Otherwise we want to biholomorphically map to a free spectrahedron (ideally two different spectrahedra)
  - `Analagous to bianalytic maps between convex sets in SCV`

Free polynomials maps

- $x = (x_1, \ldots, x_g)$ are free noncommuting variables
- $\langle x \rangle$ is the set of words in x;

$$\alpha = x_{i_1} x_{i_2} \ldots x_{i_m}$$

- $\mathbb{C}\langle x \rangle$, the **free polynomials**

$$p(x) = \sum_{\alpha \in \langle x \rangle} p_\alpha \alpha, \quad p_\alpha \in \mathbb{C}$$

- If $p \in M_d(\mathbb{C}\langle x \rangle)$ then

$$p(x) = \sum_{\alpha \in \langle x \rangle} p_\alpha \otimes x^\alpha, \quad p_\alpha \in M_d(\mathbb{C})$$

- $M(\mathbb{C}^g) = (M_n(\mathbb{C}^g))_n$
- Let $X = (X_1, X_2, \ldots, X_g) \in M(\mathbb{C})^g$ and $\alpha = x_{j_1} x_{j_2} \ldots x_{j_m} \in \langle x \rangle$
  - $X^\alpha = X_{j_1} X_{j_2} \ldots X_{j_m} = \alpha(X)$
  - $p(X) = \sum p_\alpha \otimes X^\alpha \in M_{nd}(\mathbb{C})$

Free rational and free analytic functions

- Free **rational** functions (⌚`predate free analysis`⌚)
  - $r(x_1, x_2) = (1 - x_1)^{-1} x_2 (1 - x_1)^{-1}$ is such a function (expression)
  - a free rational function has a representation

$$r(x) = c^T (I - \sum E_j x_j)^{-1} b$$

    for some $d \in \mathbb{N}$ and $b, c \in \mathbb{C}^d$ and $E \in M_d(\mathbb{C}^g)$.
  - free rational functions are evaluated on $M(\mathbb{C}^g)$ in the *obvious* way
  - Minimal realizations have no `hidden singularities`
  - See [Kaliuzhnyi-Verbovetskyi-Vinnikov], [HMV], [Klep-Volcic], [Klep-Pascoe-Volcic]
- Free analytic functions go back to at least Joe Taylor
  - See [Kaliuzhnyi-Verbovetskyi-Vinnikov] for comprehensive book

Free domains and free mappings

- We say $U \subset M(\mathbb{C}^g)$ is **free** if
  - $X \oplus Y = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \in U$ and $V^*XV \in U$, for all unitary V.
- A function f is said to be **free** if
  - $f(X \oplus Y) = f(X) \oplus f(Y)$ and $f(V^*XV) = V^*f(X)V$
- Recall a **free spectrahedron** is

$$\mathcal{D}_A = (\mathcal{D}_A(n))_n = \{X \in M(\mathbb{C})^g : L_A(X) \succeq 0\} \subset M(\mathbb{C})^g$$

- A **free polynonial (rational [analytic]) mapping** on $M(\mathbb{C})^g$ is a free map $f : M(\mathbb{C}^g) \to M(\mathbb{C}^g)$

$$f = (f_1, f_2, \ldots, f_g)$$

where each f_j is a free polynomial (rational [analytic]) map.

Free biholomorphisms and convexotonic mappings

- Free domains $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are **biholomorphic** (**bianalytic**) if there exist free analytic mappings $p : \mathcal{D} \to \tilde{\mathcal{D}}$ and $q : \tilde{\mathcal{D}} \to \mathcal{D}$ such that they are inverses on $\mathcal{D}$ and $\tilde{\mathcal{D}}$.
- A highly structured type of birational mappings are the **convextonic** mappings;
- A g-tuple of $g \times g$ matrices $\Xi = (\Xi_1, \ldots, \Xi_g)$ is **convexotonic** if

$$\Xi_k \Xi_j = \sum_{s=1}^{g} (\Xi_j)_{k,s} \Xi_s,$$

- If $\alpha \in \langle x \rangle$ then

$$\Xi_k \Xi^\alpha = \sum_{s=1}^{g} (\Xi^\alpha)_{k,s} \Xi_s,$$

- Note that Ξ spans an algebra of dimension $\leq g$.

Convexotonic too

- We say the rational mappings p and q are **convexotonic** if they are defined by

$$p_i(x) = \sum_j x_j (I - \Lambda_{\Xi}(x))^{-1}_{j,i} \qquad q_i(x) = \sum_j x_j (I + \Lambda_{\Xi}(x))^{-1}_{j,i}$$

that is $p(x) = x(I - \Lambda_{\Xi}(x))^{-1}$ and $q(x) = x(I + \Lambda_{\Xi}(x))^{-1}$

- Convexotonic maps are fundamental objects and to each map is an associated pair of `bianalytic spectrahedra`

Also convexotonic

- Suppose $(R_1, \ldots, R_g) \in M_d(\mathbb{C}^g)$ spans a g-dimensional algebra $\mathcal{R}$ with structure matrices Ξ;

$$R_k R_j = \sum_{s=1}^{g} (\Xi_j)_{k,s} R_s,$$

  and suppose C is a $d \times d$ unitary and $A \in M_d(\mathbb{C}^g)$ with the properties
  - $R_j = (C - I)A_j, 1 \leq j \leq g$
  - $A_k R_j = \sum_{s=1}^{g} (\Xi_j)_{k,s} A_s$
- We call the so constructed $(\mathcal{D}_A, \mathcal{D}_{CA})$ a **spectrahedral pair** associated with the algebra $\mathcal{R}$

Also too convexotonic

- [Theorem 1] Any spectrahedral pair $(\mathcal{D}_A, \mathcal{D}_{CA})$ with C unitary, which is associated to a g-dimensional algebra $\mathcal{R}$, has $\mathcal{D}_A$ biholomorphic to $\mathcal{D}_{CA}$ under the convexotonic mapping p whose structure matrices Ξ are associated to the algebra $\mathcal{R}$.
- [Conjecture] Up to conjugation with affine linear maps, the only bounded free spectrahedra $\mathcal{D}_A$, $\mathcal{D}_B$ which are p-bianalytic arise as spectrahedral pairs associated to an algebra $\mathcal{R}$ where p is the corresponding convexotonic mapping.

Generically true conjecture

- [Theorem 2] Suppose $A \in M_d(\mathbb{C}^g)$ and $B \in M_e(\mathbb{C}^g)$ are **generic**. If $\mathcal{D}_A$ and $\mathcal{D}_B$ are p-bianalytic with $p(0) = 0$ then
  1. d = e
  2. $\exists$ a $d \times d$ matrix C such that B is `unitarily equivalent to` CA
  3. the tuple $R = (C - I)A$ spans an algebra and,
  4. letting Ξ denote the structure matrices for the algebra, p is convexotonic with structure matrices Ξ
- The proof of this theorem takes advantage of multiple independently interesting results;
  - An analytic convex Positivstellensatz and hereditary convex Positivstellensatz
  - A one term Positivstellensatz certificate with normalization gives us the theorem (the generic condition guarantees this certificate)
  - Uniform approximation of a bounded free analytic function on a free pseudoconvex set [Agler-McCarthy]
    - A free pseudoconvex set looks like $G_Q = \{X \in M(\mathbb{C}^g) : ||Q(X)|| < 1\}$

Classifying free spectrahedra

- We want to understand what convexotonic maps arise between bianalytic spectrahedra
- If $p : \mathcal{D}_A \to \mathcal{D}_B$ is a free biholomorphism then [normalize]
  - there is an F and an affine linear map ℓ so that $\ell : \mathcal{D}_B \to \mathcal{D}_F$
  - with $\ell \circ p = \tilde{p} : \mathcal{D}_A \to \mathcal{D}_F$ is a free biholomorphism
  - $\tilde{p}(0) = 0$ and $\tilde{p}'(0) = I$.
- In particular if p is a free polynomial (rational) biholomorphism, then $\tilde{p}$ is a free polynomial (rational) biholomorphism
- With the additional assumption that A, B are **generic** and the same size we can apply [Theorem 2]:
  - So $A = \mathfrak{U}^* U B \mathfrak{U}$ for some unitaries $\mathfrak{U}, U$
  - $A_k(U - I)A_j = \sum_s (\Xi_j)_{k,s} \Xi_s$
  - $\tilde{p} = x(I - \Lambda_\Xi(x))^{-1}$

Classify!

$A = \mathfrak{U}^* UB\mathfrak{U}$, $A_k(U-I)A_j = \sum_s (\Xi_j)_{k,s}\Xi_s$, $\tilde{p} = x(I - \Lambda_\Xi(x))^{-1}$

- We want to classify the biholomorphisms between $\mathcal{D}_A$ and $\mathcal{D}_B$ by using the rich structure that we have on $\tilde{p} : \mathcal{D}_A \to \mathcal{D}_F$.
  - Things get messy quickly so we turn to a specific class of spectrahedra;
- Let Q be an invertible 2×2 matrix so that $\mathcal{D}_Q = \{c \in \mathbb{C} : I + cQ + \overline{c}Q^* \succeq 0\}$ is bounded
- Take 2×2 invertible P_{12}, P_{21}, P_{22} with $P_{21}P_{12} = \kappa Q$, $\kappa \neq 0$
- Let

$$A_1 = \begin{pmatrix} 0 & P_{12} \\ P_{21} & P_{22} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}.$$

$\mathcal{D}_Q$ is bounded, $P_{21}P_{12} = \kappa Q$, $A = \left(\begin{pmatrix} 0 & P_{12} \\ P_{21} & P_{22} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix} \right)$

★ [Example?] Suppose

- $\{Q, Q^*, P_{12}^* P_{12}, P_{21} P_{21}^*\}$ is linearly independent
- $\exists c \neq 0$ with $P_{21}^* + cP_{12}$ not invertible but $P_{21}^* - cP_{12}$ invertible.
- $B = \begin{pmatrix} \gamma I_2 & 0 \\ 0 & I_2 \end{pmatrix} A$, for a unimodular γ

1. If $\{Q, P_{22}, P_{12}^* P_{12}, P_{21} P_{21}^*\}$ is linearly independent, then
   (a) $\mathcal{D}_A$ has no non-trivial polynomial automorphisms;
   (b) if $q : \mathcal{D}_A \to \mathcal{D}_B$ is a polynomial biholomorphism, then $q(x) = (x_1, x_2 - \kappa(1-\gamma)x_1^2)$; and
   - In particular, $\mathcal{D}_A$ and $\mathcal{D}_B$ are polynomially equivalent, but **not** affine linearly equivalent

2. If $P_{22} = \alpha_1 Q + \alpha_2 Q^* + \alpha_3 P_{12}^* P_{12} + \alpha_4 P_{21} P_{21}^*$, then either
   (a) $\alpha_2 \neq 0$ and conclusion of item (1) holds; or
   (b) $\alpha_2 = 0$ and the polynomial automorphisms of $\mathcal{D}_A$ are explicitly parameterized by the unit circle.

What next?

- Continue with classification of spectrahedra
- Study the structure of convextonic mappings
  - A mapping $p(x) = x(I - \Lambda_\Gamma(x))^{-1}$, where Γ is not necessarily convexotonic has a **LFT representation**
  - Define $(\Delta_j)_{k,\ell} = (\Gamma_\ell)_{j,k}$
  - The mapping $q(x) = (I - \Lambda_\Delta(x))^{-1}x$ turns out to be the inverse of $p(x)$
  - Can we generalize this more?

Convexotonic examples

☞

$$\Xi = \left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0_3 \right)$$

✎ $p(x) = (x_1,\ x_2 + x_1x_1,\ x_3 + x_1(x_2 + x_1^2) + x_2x_1)$

✎ $p^{-1}(y) = (y_1,\ y_2 - y_1y_1,\ y_3 + y_1(y_2 - y_1^2) - y_2y_1)$

☞

$$\Xi = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

✎ $p(x) = ((1 - x_3)^{-1}x_1(1 - x_2)^{-1},\ x_2(1 - x_2)^{-1},\ x_3(1 - x_3)^{-1})$

✎ $p^{-1}(y) = ((1 + y_3)^{-1}y_1(1 + y_2)^{-1},\ (1 + y_2)^{-1}y_2,\ (1 + y_3)^{-1}y_3)$

LFT Examples (non-convexotonic)

☞

$$\Xi = \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right)$$

✎ $p(x) = (x_1 + x_2 x_1,\ x_2)$

✎ $p^{-1}(y) = ((1 - y_2)^{-1} y_1,\ y_2)$

☞

$$\Xi = \left(\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, 0_4 \right)$$

✎ $p(x) = (x_1, x_2, x_3 + x_1 x_2, x_4 + x_1^2 + x_3^2 + x_1 x_2 x_3)$

✎ $p^{-1}(y) = (y_1, y_2, y_3 - y_1 y_2, y_4 - y_1^2 - y_3^2 + y_3 y_1 y_2)$

LFT Examples (non-convexotonic)

☞

$$\Xi = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right)$$

✎ $p(x) = (x_1(1+x_2)^{-1},\ x_2(1+x_1)^{-1})$

✎ $p^{-1}(y) = (y_1 + y_1(1-y_2y_1)^{-1}y_2(1+y_1),\ y_1(1-y_2y_1)^{-1}y_2(1+y_1))$

Thank you.