Harmonic maps and shift invariant subspaces

Alexandru Aleman, University of Lund
SEAM 2017

joint with R. Pacheco and J.C. Wod

A smooth map φ between two Riemannian manifolds (M, g) and (N, h) is harmonic if it is a critical point of the energy functionals

$$E(\varphi, D) = \frac{1}{2}\int_D |d\varphi|^2 d\omega_g$$

where D is compact in M, ω_g is the volume measure and $|d\varphi|$ is the Hilbert-Schmidt norm of the differential of φ.

There is an extensive literature on the subject starting from the 60's: Eells and Sampson (1964).

- Recently I found 1607 papers on MathSci whose title contains the words " harmonic maps'.
- Harmonic diffeomorphisms from domains in $\mathbb{C}$ to $\mathbb{C}$ are currently investigated by complex analysts. In particular, they can be used to construct nontrivial particular solutions with vorticity for the Euler partial differential equations which describe the motion of a fluid.

There is an extensive literature on the subject starting from the 60's: Eells and Sampson (1964).

- Recently I found 1607 papers on MathSci whose title contains the words " harmonic maps'.
- Harmonic diffeomorphisms from domains in $\mathbb{C}$ to $\mathbb{C}$ are currently investigated by complex analysts. In particular, they can be used to construct nontrivial particular solutions with vorticity for the Euler partial differential equations which describe the motion of a fluid.

There is an extensive literature on the subject starting from the 60's: Eells and Sampson (1964).

- Recently I found 1607 papers on MathSci whose title contains the words " harmonic maps'.
- Harmonic diffeomorphisms from domains in $\mathbb{C}$ to $\mathbb{C}$ are currently investigated by complex analysts. In particular, they can be used to construct nontrivial particular solutions with vorticity for the Euler partial differential equations which describe the motion of a fluid.

In differential geometry, one of the major directions of research related to this topic concerns harmonic maps from a Riemann surface into

- compact Lie groups,
- quotients of such groups, for ex. symmetric or k-symmetric spaces

In differential geometry, one of the major directions of research related to this topic concerns harmonic maps from a Riemann surface into

- compact Lie groups,
- quotients of such groups, for ex. symmetric or k-symmetric spaces

Harmonic maps into $U(n)$

- In this context, harmonic maps from Riemann surfaces into the the group $U(n)$ of unitary $n \times n$ matrices play a central role. For example, they are equivalent to harmonic maps of finite energy from the plane, they provide a nonlinear σ-model for particle physics, and they give minimal branch immersions of S^2.
- **A great advantage is that in this context we can easily differentiate!**

Harmonic maps into $U(n)$

- In this context, harmonic maps from Riemann surfaces into the the group $U(n)$ of unitary $n \times n$ matrices play a central role. For example, they are equivalent to harmonic maps of finite energy from the plane, they provide a nonlinear σ-model for particle physics, and they give minimal branch immersions of S^2.
- **A great advantage is that in this context we can easily differentiate!**

The differential equation

A smooth map $\varphi : M \to U(n)$ with

$$\varphi^{-1} d\varphi = A dz + B d\overline{z}, \quad A = \varphi^{-1}\partial_z \varphi, B = \varphi^{-1}\partial_{\overline{z}}\varphi = -A^*,$$

is harmonic if and only if

$$\partial_{\overline{z}} A + \partial_z B = 0.$$

Examples

- Let $z \to \alpha(z),\ z \in M$ is a holomorphic subbundle of $M \times \mathbb{C}^n$, i.e. $\partial_{\bar{z}}\alpha(z) \subseteq \alpha(z)$,
- meaning that if $\sigma : M \to \mathbb{C}^n$ is smooth and satisfies $\sigma(z) \in \alpha(z)$ then $\partial_{\bar{z}}\sigma(z) \in \alpha(z)$.
- Then if π_α denotes the orthogonal projection on α, it follows easily that $\varphi(z) = \pi_{\alpha(z)} - \pi^{\perp}_{\alpha(z)}$, is a $U(n)$-valued harmonic map.
- More generally, if $\varphi : M \to \mathbf{Gr}(r, \mathbb{C}^n)$, the r-dimensional Grassmannian manifold, satisfies the original definition of harmonicity, then $\pi_{\varphi(z)} - \pi^{\perp}_{\varphi(z)}$ is a $U(n)$-valued harmonic map (Cartan embedding of φ).

Examples

- Let $z \to \alpha(z),\ z \in M$ is a holomorphic subbundle of $M \times \mathbb{C}^n$, i.e. $\partial_{\overline{z}}\alpha(z) \subseteq \alpha(z)$,
- meaning that if $\sigma : M \to \mathbb{C}^n$ is smooth and satisfies $\sigma(z) \in \alpha(z)$ then $\partial_{\overline{z}}\sigma(z) \in \alpha(z)$.
- Then if π_α denotes the orthogonal projection on α, it follows easily that $\varphi(z) = \pi_{\alpha(z)} - \pi^{\perp}_{\alpha(z)}$, is a $U(n)$-valued harmonic map.
- More generally, if $\varphi : M \to \mathbf{Gr}(r, \mathbb{C}^n)$, the r-dimensional Grassmannian manifold, satisfies the original definition of harmonicity, then $\pi_{\varphi(z)} - \pi^{\perp}_{\varphi(z)}$ is a $U(n)$-valued harmonic map (Cartan embedding of φ).

Examples

- Let $z \to \alpha(z),\ z \in M$ is a holomorphic subbundle of $M \times \mathbb{C}^n$, i.e. $\partial_{\bar{z}}\alpha(z) \subseteq \alpha(z)$,
- meaning that if $\sigma : M \to \mathbb{C}^n$ is smooth and satisfies $\sigma(z) \in \alpha(z)$ then $\partial_{\bar{z}}\sigma(z) \in \alpha(z)$.
- Then if π_α denotes the orthogonal projection on α, it follows easily that $\varphi(z) = \pi_{\alpha(z)} - \pi^{\perp}_{\alpha(z)}$, is a $U(n)$-valued harmonic map.
- More generally, if $\varphi : M \to \mathbf{Gr}(r, \mathbb{C}^n)$, the r-dimensional Grassmannian manifold, satisfies the original definition of harmonicity, then $\pi_{\varphi(z)} - \pi^{\perp}_{\varphi(z)}$ is a $U(n)$-valued harmonic map (Cartan embedding of φ).

Examples

- Let $z \to \alpha(z),\ z \in M$ is a holomorphic subbundle of $M \times \mathbb{C}^n$, i.e. $\partial_{\overline{z}}\alpha(z) \subseteq \alpha(z)$,
- meaning that if $\sigma : M \to \mathbb{C}^n$ is smooth and satisfies $\sigma(z) \in \alpha(z)$ then $\partial_{\overline{z}}\sigma(z) \in \alpha(z)$.
- Then if π_α denotes the orthogonal projection on α, it follows easily that $\varphi(z) = \pi_{\alpha(z)} - \pi^{\perp}_{\alpha(z)}$, is a $U(n)$-valued harmonic map.
- More generally, if $\varphi : M \to \mathbf{Gr}(r, \mathbb{C}^n)$, the r-dimensional Grassmannian manifold, satisfies the original definition of harmonicity, then $\pi_{\varphi(z)} - \pi^{\perp}_{\varphi(z)}$ is a $U(n)$-valued harmonic map (Cartan embedding of φ).

Extended solutions

Assume M is simply connected, let $\varphi : M \to U(n)$ harmonic with

$$\varphi^{-1} d\varphi = A dz + B d\bar{z}.$$

An *extended solution* is a function $\Phi : \mathbb{T} \times M \mapsto U(n)$ with the property that for each $\lambda \in \mathbb{T}$, $\Phi(\lambda, \cdot)$ is a global solution of the differential equation

$$\Phi^{-1}(\lambda, \cdot) d\Phi(\lambda, \cdot) = \frac{1}{2}(1 - \lambda^{-1}) A dz + \frac{1}{2}(1 - \lambda) B d\bar{z}. \qquad (1)$$

Extended solutions

- Existence is granted.
- Looks like a "cheap trick"; Harmonicity condition fulfilled by $\varphi \Leftrightarrow \partial_z \partial_{\bar{z}} \Phi = \partial_{\bar{z}} \partial_z \Phi$.
- Solutions are usually normalized by setting $\Phi(1, \cdot) = id$.
- Note that $\Phi(-1, \cdot) = u\varphi$, for some constant $u \in U(n)$
- -1 is of no importance here, we could change the equation so that we "recover" the original map at any point of the unit circle.
- The solution is not unique, but any other solution Φ_1 has the form $\Phi_1(\lambda, z) = \gamma(\lambda)\Phi_2(\lambda, z)$, with γ unitary-valued.

Extended solutions

- Existence is granted.
- Looks like a "cheap trick"; Harmonicity condition fulfilled by $\varphi \Leftrightarrow \partial_z \partial_{\bar{z}} \Phi = \partial_{\bar{z}} \partial_z \Phi$.
- Solutions are usually normalized by setting $\Phi(1, \cdot) = id$.
- Note that $\Phi(-1, \cdot) = u\varphi$, for some constant $u \in U(n)$
- -1 is of no importance here, we could change the equation so that we "recover" the original map at any point of the unit circle.
- The solution is not unique, but any other solution Φ_1 has the form $\Phi_1(\lambda, z) = \gamma(\lambda)\Phi_2(\lambda, z)$, with γ unitary-valued.

Extended solutions

- Existence is granted.
- Looks like a "cheap trick"; Harmonicity condition fulfilled by $\varphi \Leftrightarrow \partial_z \partial_{\bar{z}} \Phi = \partial_{\bar{z}} \partial_z \Phi$.
- Solutions are usually normalized by setting $\Phi(1, \cdot) = id$.
- Note that $\Phi(-1, \cdot) = u\varphi$, for some constant $u \in U(n)$
- -1 is of no importance here, we could change the equation so that we "recover" the original map at any point of the unit circle.
- The solution is not unique, but any other solution Φ_1 has the form $\Phi_1(\lambda, z) = \gamma(\lambda)\Phi_2(\lambda, z)$, with γ unitary-valued.

Extended solutions

- Existence is granted.
- Looks like a "cheap trick"; Harmonicity condition fulfilled by $\varphi \Leftrightarrow \partial_z \partial_{\overline{z}} \Phi = \partial_{\overline{z}} \partial_z \Phi$.
- Solutions are usually normalized by setting $\Phi(1, \cdot) = id$.
- Note that $\Phi(-1, \cdot) = u\varphi$, for some constant $u \in U(n)$
- -1 is of no importance here, we could change the equation so that we "recover" the original map at any point of the unit circle.
- The solution is not unique, but any other solution Φ_1 has the form $\Phi_1(\lambda, z) = \gamma(\lambda)\Phi_2(\lambda, z)$, with γ unitary-valued.

Extended solutions

- Existence is granted.
- Looks like a "cheap trick"; Harmonicity condition fulfilled by $\varphi \Leftrightarrow \partial_z \partial_{\overline{z}} \Phi = \partial_{\overline{z}} \partial_z \Phi$.
- Solutions are usually normalized by setting $\Phi(1, \cdot) = id$.
- Note that $\Phi(-1, \cdot) = u\varphi$, for some constant $u \in U(n)$
- -1 is of no importance here, we could change the equation so that we "recover" the original map at any point of the unit circle.
- The solution is not unique, but any other solution Φ_1 has the form $\Phi_1(\lambda, z) = \gamma(\lambda)\Phi_2(\lambda, z)$, with γ unitary-valued.

Extended solutions

- Existence is granted.
- Looks like a "cheap trick"; Harmonicity condition fulfilled by $\varphi \Leftrightarrow \partial_z \partial_{\overline{z}} \Phi = \partial_{\overline{z}} \partial_z \Phi$.
- Solutions are usually normalized by setting $\Phi(1, \cdot) = id$.
- Note that $\Phi(-1, \cdot) = u\varphi$, for some constant $u \in U(n)$
- -1 is of no importance here, we could change the equation so that we "recover" the original map at any point of the unit circle.
- The solution is not unique, but any other solution Φ_1 has the form $\Phi_1(\lambda, z) = \gamma(\lambda)\Phi_2(\lambda, z)$, with γ unitary-valued.

Nothing is explicit here!

- In the context described before, finding harmonic maps from the harmonicity condition is hopeless!

- Finding explicit extended solutions is hopeless as well!

- There is a rich qualitative theory of such maps containing a number of interesting questions from the analytic point of view.

Nothing is explicit here!

- In the context described before, finding harmonic maps from the harmonicity condition is hopeless!
- Finding explicit extended solutions is hopeless as well!
- There is a rich qualitative theory of such maps containing a number of interesting questions from the analytic point of view.

Nothing is explicit here!

- In the context described before, finding harmonic maps from the harmonicity condition is hopeless!
- Finding explicit extended solutions is hopeless as well!
- There is a rich qualitative theory of such maps containing a number of interesting questions from the analytic point of view.

Extended framings

- There are many alternatives to the equations defining extended solutions, which yield smooth maps $\Psi : \mathbb{T} \times M \to \mathrm{GL}(n, \mathbb{C})$.
- They are called *extended framings*.
- Main disadvantage: Take us out of the framework of Lie groups! Nontrivial factorizations needed to bring them back.

Extended framings

- There are many alternatives to the equations defining extended solutions, which yield smooth maps $\Psi : \mathbb{T} \times M \to \mathrm{GL}(n, \mathbb{C})$.
- They are called *extended framings*.
- Main disadvantage: Take us out of the framework of Lie groups! Nontrivial factorizations needed to bring them back.

Extended framings

- There are many alternatives to the equations defining extended solutions, which yield smooth maps $\Psi : \mathbb{T} \times M \to \mathrm{GL}(n, \mathbb{C})$.
- They are called *extended framings*.
- Main disadvantage: Take us out of the framework of Lie groups! Nontrivial factorizations needed to bring them back.

An example of extended framing

An important class of harmonic maps (of finite type) possess extended framings of the form

$$\Psi(\lambda, z) = \exp(\lambda^{-1}\tau(z)p(\lambda)),$$

where p is a polynomial with coefficients in $\mathbb{M}_{n\times n}(\mathbb{C})$, independent of $z \in M$, $p(0) \neq 0$ and $\tau : M \to \mathbb{C}$ is analytic.

Uhlenbeck's approach

K. Uhlenbeck ('89) showed that if $M = S^2$ then:
The extended solutions are essentially **Blaschke-Potapov products** depending on $z \in M$, i.e.

$$\Phi(\lambda, z) = \gamma(\lambda) \prod_{j=1}^{m} (\pi_{\alpha_j(z)} + \lambda \pi^{\perp}_{\alpha_j(z)}),$$

where $\alpha_1, \ldots, \alpha_m$ are subbundles of $M \times \mathbb{C}^n$, and $\pi_\alpha, \pi_\alpha^\perp$ denote the orthogonal projections on $\alpha, \alpha^\perp$.

Finite uniton number

Let M be a simply connected Riemann surface (compact or not) and let $\varphi : M \to U(n)$ be a harmonic map. If one (and hence all) extended solution Φ has the form

$$\Phi(\lambda, z) = \gamma(\lambda) \prod_{j=1}^{m} (\pi_{\alpha_j(z)} + \lambda \pi^{\perp}_{\alpha_j(z)})$$

with differentiable subbundles of $M \times \mathbb{C}^n$, $\alpha_1, \dots, \alpha_m$, we say that φ has *finite uniton number*.

Unitons

- If $n > 1$, there are several ways to factor the same Blaschke-Potapov product.
- For extended solutions it turns out that the product can be factored in such a way that for each $k \leq m$

$$\Phi_k(\lambda, z) = \gamma(\lambda) \prod_{j=k}^{m} (\pi_{\alpha_j(z)} + \lambda \pi^{\perp}_{\alpha_j(z)}),$$

is an extended solution as well.
- This was first done by Uhlenbbeck, and then Svensson and Wood (2012) described all such factorizations. The factors $\pi_{\alpha_j(z)} - \pi^{\perp}_{\alpha_j(z)}$ are called *unitons of* φ.

Unitons

- If $n > 1$, there are several ways to factor the same Blaschke-Potapov product.
- For extended solutions it turns out that the product can be factored in such a way that for each $k \leq m$

$$\Phi_k(\lambda, z) = \gamma(\lambda) \prod_{j=k}^{m} (\pi_{\alpha_j(z)} + \lambda \pi^{\perp}_{\alpha_j(z)}),$$

  is an extended solution as well.
- This was first done by Uhlenbbeck, and then Svensson and Wood (2012) described all such factorizations. The factors $\pi_{\alpha_j(z)} - \pi^{\perp}_{\alpha_j(z)}$ are called *unitons of* φ.

Unitons

- If $n > 1$, there are several ways to factor the same Blaschke-Potapov product.
- For extended solutions it turns out that the product can be factored in such a way that for each $k \leq m$

$$\Phi_k(\lambda, z) = \gamma(\lambda) \prod_{j=k}^{m} (\pi_{\alpha_j(z)} + \lambda \pi^{\perp}_{\alpha_j(z)}),$$

  is an extended solution as well.
- This was first done by Uhlenbbeck, and then Svensson and Wood (2012) described all such factorizations. The factors $\pi_{\alpha_j(z)} - \pi^{\perp}_{\alpha_j(z)}$ are called *unitons of* φ.

Uniton factorization of the original map

- The original harmonic map $\varphi : M \to U(n)$ has the form

$$\varphi = u \prod_{j=1}^{m} (\pi_{\alpha_j(z)} - \pi^{\perp}_{\alpha_j(z)}),$$

and for $1 \leq k \leq m$,

$$\varphi_k = u \prod_{j=k}^{m} (\pi_{\alpha_j(z)} - \pi^{\perp}_{\alpha_j(z)}),$$

is harmonic.

- Roughly speaking, this means that certain transformations (called Bäcklund transformations in PDE language) solve these complicated equations.

Uniton factorization of the original map

- The original harmonic map $\varphi : M \to U(n)$ has the form

$$\varphi = u \prod_{j=1}^{m} (\pi_{\alpha_j(z)} - \pi^{\perp}_{\alpha_j(z)}),$$

  and for $1 \leq k \leq m$,

$$\varphi_k = u \prod_{j=k}^{m} (\pi_{\alpha_j(z)} - \pi^{\perp}_{\alpha_j(z)}),$$

  is harmonic.
- Roughly speaking, this means that certain transformations (called Bäcklund transformations in PDE language) solve these complicated equations.

Shift- invariant subspaces, or the Grassmannian model

Let $H^2(\mathbb{C}^n)$ be the usual (vector-valued) Hardy space, i.e., the closed subspace of $L^2(\mathbb{T}, \mathbb{C}^n)$ consisting of Fourier series whose negative coefficients vanish.

Let $\varphi : M \to U(n)$ be a harmonic map and let Φ be an extended solution.
Set

$$W_z = \Phi(\cdot, z)H^2(\mathbb{C}^n)$$

This is a closed subspace invariant for the forward shift $S|L^2(\mathbb{T}, \mathbb{C}^n)$,

$$Sf(\lambda) = \lambda f(\lambda).$$

Shift- invariant subspaces, or the Grassmannian model

Let $H^2(\mathbb{C}^n)$ be the usual (vector-valued) Hardy space, i.e., the closed subspace of $L^2(\mathbb{T}, \mathbb{C}^n)$ consisting of Fourier series whose negative coefficients vanish.

Let $\varphi : M \to U(n)$ be a harmonic map and let Φ be an extended solution.
Set

$$W_z = \Phi(\cdot, z) H^2(\mathbb{C}^n)$$

This is a closed subspace invariant for the forward shift $S|L^2(\mathbb{T}, \mathbb{C}^n)$,

$$Sf(\lambda) = \lambda f(\lambda).$$

Shift- invariant subspaces, or the Grassmannian model

- This family of subspaces gives an infinite dimensional bundle W over M with the properties

$$S\partial_z W \subset W, \quad \partial_{\bar{z}} W \subset W.$$

- i.e., if $f(\cdot, z)$ is differentiable and $f(\cdot, z) \in W_z$ for all z, then

$$S\partial_z f(\cdot, z) \in W_z, \quad \partial_{\bar{z}} f(\cdot, z) \in W_z.$$

Shift- invariant subspaces, or the Grassmannian model

- This family of subspaces gives an infinite dimensional bundle W over M with the properties

$$S\partial_z W \subset W, \quad \partial_{\overline{z}} W \subset W.$$

- i.e., if $f(\cdot, z)$ is differentiable and $f(\cdot, z) \in W_z$ for all z, then

$$S\partial_z f(\cdot, z) \in W_z, \quad \partial_{\overline{z}} f(\cdot, z) \in W_z.$$

This is trivial because if $f(\cdot, z) = \Phi(\cdot, z)g(\cdot, z)$ the product rule yields

$$S\partial_z f(\cdot, z) = \Phi(\cdot, z)[S(1 - S^{-1})A(z)f(\cdot, z) + S\partial_z f(\cdot, z)] \in W_z,$$

and

$$S\partial_{\overline{z}} f(\cdot, z) = \Phi(\cdot, z)[(1 - S)B(z)f(\cdot, z) + \partial_{\overline{z}} f(\cdot, z)] \in W_z,$$

The advantage of using W's

- Coordinate-free model, the derivatives of coordinates (chain rule) are absorbed in the subspace,
- They are essentially unique.
- They are essentially "universal".

The advantage of using W's

- Coordinate-free model, the derivatives of coordinates (chain rule) are absorbed in the subspace,

- They are essentially unique.

- They are essentially ”universal”.

The advantage of using W's

- Coordinate-free model, the derivatives of coordinates (chain rule) are absorbed in the subspace,
- They are essentially unique.
- They are essentially "universal".

The main issue:

Relate the geometric properties of a harmonic maps to the functional-analytic structure of the corresponding shift-invariant subspaces

In particular,

- Finite and infinite uniton number.
- Harmonic maps into k-symmetric spaces.
- Deformations of harmonic maps with finite uniton number.

In particular,

- Finite and infinite uniton number.
- Harmonic maps into k-symmetric spaces.
- Deformations of harmonic maps with finite uniton number.

In particular,

- Finite and infinite uniton number.
- Harmonic maps into k-symmetric spaces.
- Deformations of harmonic maps with finite uniton number.

A criterion for finite uniton number

Recall that φ has finite uniton number if there exists an extended solution Φ such that

$$\Phi(\lambda, z) = \gamma(\lambda) \prod_{j=1}^{m} (\pi_{\alpha_j(z)} + \lambda \pi^{\perp}_{\alpha_j(z)}),$$

where $\alpha_1, \ldots, \alpha_m$ are subbundles of $M \times \mathbb{C}^n$, and $\pi_\alpha, \pi_\alpha^\perp$ denote the orthogonal projections on $\alpha, \alpha^\perp$.

A criterion for finite uniton number

- Let $\varphi : M \to U(n)$ harmonic, $\Phi =$ extended solution,

$$W_z = \Phi(\cdot, z)H^2(\mathbb{C}^n).$$

- Set

$$W_z^1 = \partial_z W_z + W_z$$

- Note that

$$\partial_z \Phi f = (1 - \overline{\lambda})\Phi A f + \Phi \partial_z f = \Phi(\overline{\lambda} A f_0 + g),$$

where f_0 is constant in λ, and $g \in H^2(\mathbb{C}^n)$, i.e.

$$W_z^1 = \Phi(\overline{\lambda} \times \text{something in range } A + H^2(\mathbb{C}^n)).$$

A criterion for finite uniton number

- Let $\varphi : M \to U(n)$ harmonic, $\Phi =$ extended solution,

$$W_z = \Phi(\cdot, z)H^2(\mathbb{C}^n).$$

- Set

$$W_z^1 = \partial_z W_z + W_z$$

- Note that

$$\partial_z \Phi f = (1 - \overline{\lambda})\Phi A f + \Phi \partial_z f = \Phi(\overline{\lambda} A f_0 + g),$$

where f_0 is constant in λ, and $g \in H^2(\mathbb{C}^n)$, i.e.

$$W_z^1 = \Phi(\overline{\lambda} \times \text{something in range } A + H^2(\mathbb{C}^n)).$$

A criterion for finite uniton number

- Let $\varphi : M \to U(n)$ harmonic, $\Phi =$ extended solution,

$$W_z = \Phi(\cdot, z)H^2(\mathbb{C}^n).$$

- Set

$$W_z^1 = \partial_z W_z + W_z$$

- Note that

$$\partial_z \Phi f = (1 - \overline{\lambda})\Phi A f + \Phi \partial_z f = \Phi(\overline{\lambda} A f_0 + g),$$

where f_0 is constant in λ, and $g \in H^2(\mathbb{C}^n)$, i.e.

$$W_z^1 = \Phi(\overline{\lambda} \times \text{something in range } A + H^2(\mathbb{C}^n)).$$

continue...

- Set

$$W_z^{k+1} = \partial_z W_z^k + W_z^k = \Phi(\text{polynomial in } \overline{\lambda} + H^2(\mathbb{C}^n)).$$

Proposition

Either there exists k such that $W_z^k = W_z^{k+1}$, in which case the chain stabilizes and we have a finite uniton number, or

$$\overline{\bigvee_{k \geq o} W^k} = L^2(\mathbb{T}, \mathbb{C}^n)$$

and the uniton number is infinite.

continue...

- Set

$$W_z^{k+1} = \partial_z W_z^k + W_z^k = \Phi(\text{polynomial in } \overline{\lambda} + H^2(\mathbb{C}^n)).$$

Proposition

Either there exists k such that $W_z^k = W_z^{k+1}$, in which case the chain stabilizes and we have a finite uniton number, or

$$\overline{\bigvee_{k \geq o} W^k} = L^2(\mathbb{T}, \mathbb{C}^n)$$

and the uniton number is infinite.

A characterization of maps with finite uniton number

Theorem

For a harmonic map $\varphi : M \to U(n)$, with $\varphi^{-1}\partial_z\varphi = 2A$, the following are equivalent:
(i) φ has finite uniton number,
(ii) There exists γ unitary-valued a.e. on S^1 and independent of $z \in M$ such that $W \subset \gamma H^2(\mathbb{C}^n)$,
(iii) The sequence $(W_z^k)_{k\geq 0}$ stabilizes,
(iv) If $T(\mu),\ \mu \in \mathbb{C}$, denotes the operator on $C^\infty(M)$ defined by $T(\mu) = \partial_z + \mu A$, then the degree of the operator-valued polynomials in μ, $T^r(\mu)$, stays bounded when $r \in \mathbb{N}$.

- Note that if (iv) holds, i.e. $T^r = ((\mu A + \partial_z)^r$ has bounded degree in μ, then $A^m = 0$ for some m, i.e. A must be nilpotent.
- We say φ is *nilconformal*. However, the condition is far from sufficient!
- In fact, we can prove:

Corollary

If M is not compact there exist nilconformal harmonic maps $\varphi : M \to U(n)$ with infinite uniton number.

- Note that if (iv) holds, i.e. $T^r = ((\mu A + \partial_z)^r$ has bounded degree in μ, then $A^m = 0$ for some m, i.e. A must be nilpotent.
- We say φ is *nilconformal*. However, the condition is far from sufficient!
- In fact, we can prove:

Corollary

If M is not compact there exist nilconformal harmonic maps $\varphi : M \to U(n)$ with infinite uniton number.

- Note that if (iv) holds, i.e. $T^r = ((\mu A + \partial_z)^r$ has bounded degree in μ, then $A^m = 0$ for some m, i.e. A must be nilpotent.
- We say φ is *nilconformal*. However, the condition is far from sufficient!
- In fact, we can prove:

Corollary

If M is not compact there exist nilconformal harmonic maps $\varphi : M \to U(n)$ with infinite uniton number.

- Note that if (iv) holds, i.e. $T^r = ((\mu A + \partial_z)^r$ has bounded degree in μ, then $A^m = 0$ for some m, i.e. A must be nilpotent.
- We say φ is *nilconformal*. However, the condition is far from sufficient!
- In fact, we can prove:

Corollary

If M is not compact there exist nilconformal harmonic maps $\varphi : M \to U(n)$ with infinite uniton number.

Constant holomorphic potentials

- Assume that φ has an extended framing of the form
$$\Psi(\lambda, z) = \exp(\lambda^{-1}\tau(z)p(\lambda)),$$
where p is a polynomial with coefficients in $\mathbb{M}_{n\times n}(\mathbb{C})$, independent of $z \in M$, $p(0) \neq 0$, and $\tau : M \to \mathbb{C}$ is analytic.
- Set $W_z = \Psi(\cdot, z)H^2(\mathbb{C}^n)$. Then it follows easily that
$$W_z^k = \Psi(\cdot, z)\left(\left(\frac{p(\lambda)}{\lambda}\right)^k f_k + \ldots + \frac{p(\lambda)}{\lambda} f_1 + H^2(\mathbb{C}^n)\right),$$
with $f_1, \ldots, f_k$ sections (depend smoothly on $z \in M$).
- (W_z^k) stabilizes if and only if the order of the pole of $\left(\frac{p(\lambda)}{\lambda}\right)^k$ at the origin stays bounded when $k \in \mathbb{N}$.

Constant holomorphic potentials

- Assume that φ has an extended framing of the form

$$\Psi(\lambda, z) = \exp(\lambda^{-1}\tau(z)p(\lambda)),$$

where p is a polynomial with coefficients in $\mathbb{M}_{n\times n}(\mathbb{C})$, independent of $z \in M$, $p(0) \neq 0$, and $\tau : M \to \mathbb{C}$ is analytic.

- Set $W_z = \Psi(\cdot, z)H^2(\mathbb{C}^n)$. Then it follows easily that

$$W_z^k = \Psi(\cdot, z)\left(\left(\frac{p(\lambda)}{\lambda}\right)^k f_k + \ldots + \frac{p(\lambda)}{\lambda} f_1 + H^2(\mathbb{C}^n)\right),$$

with $f_1, \ldots, f_k$ sections (depend smoothly on $z \in M$).

- (W_z^k) stabilizes if and only if the order of the pole of $\left(\frac{p(\lambda)}{\lambda}\right)^k$ at the origin stays bounded when $k \in \mathbb{N}$.

Constant holomorphic potentials

- Assume that φ has an extended framing of the form

$$\Psi(\lambda, z) = \exp(\lambda^{-1}\tau(z)p(\lambda)),$$

  where p is a polynomial with coefficients in $\mathbb{M}_{n\times n}(\mathbb{C})$, independent of $z \in M$, $p(0) \neq 0$, and $\tau : M \to \mathbb{C}$ is analytic.
- Set $W_z = \Psi(\cdot, z)H^2(\mathbb{C}^n)$. Then it follows easily that

$$W_z^k = \Psi(\cdot, z)\left(\left(\frac{p(\lambda)}{\lambda}\right)^k f_k + \ldots + \frac{p(\lambda)}{\lambda} f_1 + H^2(\mathbb{C}^n)\right),$$

  with $f_1, \ldots, f_k$ sections (depend smoothly on $z \in M$).
- (W_z^k) stabilizes if and only if the order of the pole of $\left(\frac{p(\lambda)}{\lambda}\right)^k$ at the origin stays bounded when $k \in \mathbb{N}$.

Constant holomorphic potentials

Theorem

Assume φ *has an extended framing of the form*

$$\Psi(\lambda, z) = \exp(\lambda^{-1}\tau(z)p(\lambda)),$$

with p, τ *as above. Then* φ *has finite uniton number if and only if* $\det(\frac{p(\lambda)}{\lambda} + \mu I)$ *is a polynomial in* λ, μ*, or equivalently, for each fixed* $\mu \in \mathbb{C}$, $\det(\frac{p(\lambda)}{\lambda} + \mu I)$ *is bounded near the origin.*

Constant holomorphic potentials

- The condition is further equivalent to: The eigenvalues of $\lambda^{-1}p(\lambda)$ stay bounded when $\lambda \to 0$.
- For example when $\lambda^{-1}p(\lambda) = C\lambda^{-1} + D$ the condition says that the eigenvalues of are constant. In particular, they equal those of D.

Constant holomorphic potentials

- The condition is further equivalent to: The eigenvalues of $\lambda^{-1}p(\lambda)$ stay bounded when $\lambda \to 0$.
- For example when $\lambda^{-1}p(\lambda) = C\lambda^{-1} + D$ the condition says that the eigenvalues of are constant. In particular, they equal those of D.

k-symmetric space

- is a Riemannian manifold N with the property that for each point $p \in N$ there exists an isometry $\eta : N \to N$ with $\eta(p) = p$ and $(d\eta(p))^k = id$. Here $k \geq 2$ is an integer.
- There is an equivalent, more complicated algebraic definition where k-symmetric spaces appear as quotients of Lie groups G/K. Here G is compact ($\sim$ subgroup of $U(n)$) and K is a subgroup with certain properties.

k-symmetric space

- is a Riemannian manifold N with the property that for each point $p \in N$ there exists an isometry $\eta : N \to N$ with $\eta(p) = p$ and $(d\eta(p))^k = id$. Here $k \geq 2$ is an integer.
- There is an equivalent, more complicated algebraic definition where k-symmetric spaces appear as quotients of Lie groups G/K. Here G is compact ($\sim$ subgroup of $U(n)$) and K is a subgroup with certain properties.

Harmonic maps into k-symmetric spaces

- Thinking in terms of quotients G/K Dorfmeister, Pedit, Wu (1998) produced for a (primitive) harmonic map $\varphi : M \to G/K$, harmonic lifts $F : M \to \mathrm{GL}(n, \mathbb{C})$ as well as extended framings $F : \mathbb{T} \times M \to \mathrm{GL}(n, \mathbb{C})$.
- Pacheco (2005) considered pairs of such maps $\varphi : M \to G/K, \tilde{\varphi} : M \to G/\tilde{K}$ where G/K is k-symmetric , and $G/\tilde{K}$ is $\tilde{k}$-symmetric ($K \subset \tilde{K}$) and used an algebraic construction to exhibit an isomorphism that sends extended framings of φ to extended framings of $\tilde{\varphi}$ to

Harmonic maps into k-symmetric spaces

- Thinking in terms of quotients G/K Dorfmeister, Pedit, Wu (1998) produced for a (primitive) harmonic map $\varphi : M \to G/K$, harmonic lifts $F : M \to \mathrm{GL}(n, \mathbb{C})$ as well as extended framings $F : \mathbb{T} \times M \to \mathrm{GL}(n, \mathbb{C})$.
- Pacheco (2005) considered pairs of such maps $\varphi : M \to G/K, \tilde{\varphi} : M \to G/\tilde{K}$ where G/K is k-symmetric , and $G/\tilde{K}$ is $\tilde{k}$-symmetric ($K \subset \tilde{K}$) and used an algebraic construction to exhibit an isomorphism that sends extended framings of φ to extended framings of $\tilde{\varphi}$ to

Where are extended framings, there are $W's$

It turns out that the W's corresponding to a (primitive) harmonic map into a k-symmetric space can be chosen to satisfy

$$f \in W_z \Longrightarrow f_\omega \in W_z$$

where $\omega = e^{\frac{2\pi i}{k}}$, and $f_\omega(\lambda) = f(\omega\lambda)$.

- A shift-invariant subspace $W = \Phi H^2(\mathbb{C}^n)$ has this property if and only if

$$W = \bigoplus_{j=0}^{k-1} S^j \{g(\lambda^k) : \; g \in V_j\},$$

- where V_j are shift invariant,

$$V_0 \subseteq \ldots \subseteq V_{k-1}, \quad SV_{k-1} \subseteq V_0.$$

- In fact,

$$S^j\{g(\lambda^k) : \; g \in V_j\} = \{f \in W : \; f_\omega = \omega^j f\}$$
$$= \{h \in W : \; h(\lambda) = \sum_{l=0}^{k-1} \omega^{-lj} f(\omega^l \lambda), \; f \in W\},$$

- A shift-invariant subspace $W = \Phi H^2(\mathbb{C}^n)$ has this property if and only if

$$W = \bigoplus_{j=0}^{k-1} S^j \{g(\lambda^k) : \; g \in V_j\},$$

- where V_j are shift invariant,

$$V_0 \subseteq \ldots \subseteq V_{k-1}, \quad SV_{k-1} \subseteq V_0.$$

- In fact,

$$S^j\{g(\lambda^k) : \; g \in V_j\} = \{f \in W : \; f_\omega = \omega^j f\}$$
$$= \{h \in W : \; h(\lambda) = \sum_{l=0}^{k-1} \omega^{-lj} f(\omega^l \lambda), \; f \in W\},$$

- A shift-invariant subspace $W = \Phi H^2(\mathbb{C}^n)$ has this property if and only if

$$W = \bigoplus_{j=0}^{k-1} S^j\{g(\lambda^k) : \; g \in V_j\},$$

- where V_j are shift invariant,

$$V_0 \subseteq \ldots \subseteq V_{k-1}, \quad SV_{k-1} \subseteq V_0.$$

- In fact,

$$\begin{aligned} S^j\{g(\lambda^k) : \; g \in V_j\} &= \{f \in W : \; f_\omega = \omega^j f\} \\ &= \{h \in W : \; h(\lambda) = \sum_{l=0}^{k-1} \omega^{-lj} f(\omega^l \lambda), \; f \in W\}, \end{aligned}$$

- It turns out that $V_{k-1} = \Psi H^2(\mathbb{C}^n)$ with Ψ unitary-valued a.e. and

$$V_j = \Psi(\pi_{\alpha_j} + S\pi_{\alpha_j}^{\perp})H^2(\mathbb{C}^n),$$

where $\alpha_0 \subseteq \ldots \subseteq \alpha_{k-2}$ are subspaces of $\mathbb{C}^n$.

- Then

$$W = \Psi(\lambda^k)\prod_{j=0}^{k-2}(\pi_{\alpha_j} + \lambda\pi_{\alpha_j}^{\perp})H^2(\mathbb{C}^n)$$
$$= \Psi(\lambda^k)(\alpha_0 + \lambda\alpha_1 + \ldots + \lambda^{k-2}\alpha_{k-2} + \lambda^{k-1}H^2(\mathbb{C}^n)).$$

- It turns out that $V_{k-1} = \Psi H^2(\mathbb{C}^n)$ with Ψ unitary-valued a.e. and
$$V_j = \Psi(\pi_{\alpha_j} + S\pi_{\alpha_j}^{\perp})H^2(\mathbb{C}^n),$$
where $\alpha_0 \subseteq \ldots \subseteq \alpha_{k-2}$ are subspaces of $\mathbb{C}^n$.
- Then
$$\begin{aligned} W &= \Psi(\lambda^k) \prod_{j=0}^{k-2} (\pi_{\alpha_j} + \lambda\pi_{\alpha_j}^{\perp})H^2(\mathbb{C}^n) \\ &= \Psi(\lambda^k)(\alpha_0 + \lambda\alpha_1 + \ldots + \lambda^{k-2}\alpha_{k-2} + \lambda^{k-1}H^2(\mathbb{C}^n)). \end{aligned}$$

For extended solutions Φ:

- The conditions $S\partial_z W_z \subset W_z, \ \partial_{\bar{z}} W_z \subset W_z$ are equivalent to

$$\partial_z V_j \subset V_{j+1}, \ j < k-1, \ S\partial_z V_{k-1} \subset V_0, \ \partial_{\bar{z}} V_j \subset V_j.$$

- In particular, $\Psi, \ \Psi(\pi_{\alpha_j} + S\pi_{\alpha_j}^{\perp})$ are extended solutions as well.
- Ψ is a very special one. The harmonic map $\psi = \Psi(-1, \cdot)$ is strongly conformal in the sense that $(\psi^{-1}\partial_z\psi)^2 = 0$. Moreover, the α_j's yield special unitons for ψ (both basic and anti-basic).
- The construction can be carried out for extended framings as well with essentially the same proof. In both cases Ψ can be computed with help of Φ!

For extended solutions Φ:

- The conditions $S\partial_z W_z \subset W_z,\ \partial_{\bar{z}} W_z \subset W_z$ are equivalent to

$$\partial_z V_j \subset V_{j+1},\ j < k-1,\ S\partial_z V_{k-1} \subset V_0,\ \partial_{\bar{z}} V_j \subset V_j.$$

- In particular, $\Psi,\ \Psi(\pi_{\alpha_j} + S\pi_{\alpha_j}^{\perp})$ are extended solutions as well.
- Ψ is a very special one. The harmonic map $\psi = \Psi(-1, \cdot)$ is strongly conformal in the sense that $(\psi^{-1}\partial_z\psi)^2 = 0$. Moreover, the α_j's yield special unitons for ψ (both basic and anti-basic).
- The construction can be carried out for extended framings as well with essentially the same proof. In both cases Ψ can be computed with help of Φ!

For extended solutions Φ:

- The conditions $S\partial_z W_z \subset W_z,\ \partial_{\bar{z}} W_z \subset W_z$ are equivalent to

$$\partial_z V_j \subset V_{j+1},\ j < k-1,\ S\partial_z V_{k-1} \subset V_0,\ \partial_{\bar{z}} V_j \subset V_j.$$

- In particular, $\Psi,\ \Psi(\pi_{\alpha_j} + S\pi^{\perp}_{\alpha_j})$ are extended solutions as well.
- Ψ is a very special one. The harmonic map $\psi = \Psi(-1,\cdot)$ is strongly conformal in the sense that $(\psi^{-1}\partial_z\psi)^2 = 0$. Moreover, the α_j's yield special unitons for ψ (both basic and anti-basic).
- The construction can be carried out for extended framings as well with essentially the same proof. In both cases Ψ can be computed with help of Φ!

For extended solutions Φ:

- The conditions $S\partial_z W_z \subset W_z,\ \partial_{\bar z} W_z \subset W_z$ are equivalent to

$$\partial_z V_j \subset V_{j+1},\ j < k-1,\ S\partial_z V_{k-1} \subset V_0,\ \partial_{\bar z} V_j \subset V_j.$$

- In particular, $\Psi,\ \Psi(\pi_{\alpha_j} + S\pi_{\alpha_j}^{\perp})$ are extended solutions as well.
- Ψ is a very special one. The harmonic map $\psi = \Psi(-1,\cdot)$ is strongly conformal in the sense that $(\psi^{-1}\partial_z\psi)^2 = 0$. Moreover, the α_j's yield special unitons for ψ (both basic and anti-basic).
- The construction can be carried out for extended framings as well with essentially the same proof. In both cases Ψ can be computed with help of Φ!

Corollary

Let φ be a harmonic map into a k-symmetric space and let $\Phi,\ \Psi, \alpha_j$ as above. For $2 \leq s < k$ and $0 \leq j_0 < j_1 < \ldots < j_{s-2} \leq k-2$,

$$\Phi_s(\lambda, z) = \Psi(\lambda^s) \prod_{l=0}^{s-2} (\pi_{\alpha_{j_l}} + \lambda \pi_{\alpha_{j_l}}^{\perp})$$

is an extended solution (framing) for a harmonic map ψ into a s-symmetric space.

Theorem

(i) Φ has finite uniton number if and only if Ψ has.
(ii) For extended framings Ψ is of finite type if and only if Ψ is of finite type.

Corollary

Let φ be a harmonic map into a k-symmetric space and let $\Phi,\ \Psi, \alpha_j$ as above. For $2 \leq s < k$ and $0 \leq j_0 < j_1 < \ldots < j_{s-2} \leq k-2$,

$$\Phi_s(\lambda, z) = \Psi(\lambda^s) \prod_{l=0}^{s-2} (\pi_{\alpha_{j_l}} + \lambda \pi_{\alpha_{j_l}}^{\perp})$$

is an extended solution (framing) for a harmonic map ψ into a s-symmetric space.

Theorem

(i) Φ has finite uniton number if and only if Ψ has.
(ii) For extended framings Ψ is of finite type if and only if Ψ is of finite type.

Corollary

Let φ be a harmonic map into a k-symmetric space and let $\Phi,\ \Psi, \alpha_j$ as above. For $2 \leq s < k$ and $0 \leq j_0 < j_1 < \ldots < j_{s-2} \leq k-2$,

$$\Phi_s(\lambda, z) = \Psi(\lambda^s) \prod_{l=0}^{s-2} (\pi_{\alpha_{j_l}} + \lambda \pi^{\perp}_{\alpha_{j_l}})$$

is an extended solution (framing) for a harmonic map ψ into a s-symmetric space.

Theorem

(i) Φ has finite uniton number if and only if Ψ has.
(ii) For extended framings Ψ is of finite type if and only if Ψ is of finite type.

Deformations: Joint work with M.J. Martinéz, A.M. Persson, M. Svensson.

- A class of harmonic maps which is quite well understood consists of those which are S^1-*invariant*, i.e.there is an extended solution Φ with

$$\Phi(\lambda_1\lambda_2, z) = \phi(\lambda_1, z)\Phi(\lambda_2, z), \quad \lambda_1, \lambda_2 \in \mathbb{T},\ z \in M,$$

or equivalently, the W's are invariant under composition with rotations.

- Assume that the extended solution extends analytically to the unit disc.. Such maps have finite uniton number!

- **Can we deform continuously such a map φ into an S^1-invariant one?**

Deformations: Joint work with M.J. Martinéz, A.M. Persson, M. Svensson.

- A class of harmonic maps which is quite well understood consists of those which are S^1-*invariant*, i.e.there is an extended solution Φ with

$$\Phi(\lambda_1\lambda_2, z) = \phi(\lambda_1, z)\Phi(\lambda_2, z), \quad \lambda_1, \lambda_2 \in \mathbb{T},\ z \in M,$$

  or equivalently, the W's are invariant under composition with rotations.

- Assume that the extended solution extends analytically to the unit disc.. Such maps have finite uniton number!

- **Can we deform continuously such a map φ into an S^1-invariant one?**

Deformations: Joint work with M.J. Martinéz, A.M. Persson, M. Svensson.

- A class of harmonic maps which is quite well understood consists of those which are S^1-*invariant*, i.e.there is an extended solution Φ with

$$\Phi(\lambda_1\lambda_2, z) = \phi(\lambda_1, z)\Phi(\lambda_2, z), \quad \lambda_1, \lambda_2 \in \mathbb{T}, \ z \in M,$$

  or equivalently, the W's are invariant under composition with rotations.
- Assume that the extended solution extends analytically to the unit disc.. Such maps have finite uniton number!

- **Can we deform continuously such a map φ into an S^1-invariant one?**

Sure!

- We dilate Φ, i.e. we consider $\Phi(\mu\lambda, z),\ 0 < |\mu| < 1$ and factor it

$$\Phi(\mu\lambda, z) = \text{Blaschke-Potapov} \times \text{outer}$$

- The Blaschke-Potapov factor Φ^{μ} should be an extended solution by the "universality" of the W's.
- If

$$\lim_{\mu \to 0} \Phi^{\mu} \text{ exists,}$$

it should be S^1-invariant.

Sure!

- We dilate Φ, i.e. we consider $\Phi(\mu\lambda, z), \ 0 < |\mu| < 1$ and factor it

$$\Phi(\mu\lambda, z) = \text{Blaschke-Potapov} \times \text{outer}$$

- The Blaschke-Potapov factor Φ^μ should be an extended solution by the "universality" of the W's.
- If

$$\lim_{\mu \to 0} \Phi^\mu \ \text{exists,}$$

it should be S^1-invariant.

Sure!

- We dilate Φ, i.e. we consider $\Phi(\mu\lambda, z),\ 0 < |\mu| < 1$ and factor it

$$\Phi(\mu\lambda, z) = \text{Blaschke-Potapov} \times \text{outer}$$

- The Blaschke-Potapov factor Φ^μ should be an extended solution by the "universality" of the W's.
- If

$$\lim_{\mu\to 0} \Phi^\mu \text{ exists,}$$

it should be S^1-invariant.

Can we find a deformation such that the unitons of φ also deform continuously?

This is a more subtle question because the standard factorization factorization Blaschke-Potapov $\times$ outer is essentially unique, but the factors of the Blaschke-Potapov product are not, and not all of its factorizations yield unitons.

Can we find a deformation such that the unitons of φ also deform continuously?

This is a more subtle question because the standard factorization factorization Blaschke-Potapov $\times$ outer is essentially unique, but the factors of the Blaschke-Potapov product are not, and not all of its factorizations yield unitons.

Theorem

Let $\varphi : M \to U(n)$ be a harmonic map with analytic extended solution Φ, and let Φ^μ be its deformation obtained by dilation of Φ

(i) $\lim_{\mu \to 0} \Phi^\mu$ exists and is an extended solution corresponding to an S^1-invariant harmonic map.

(ii) Given any uniton factorization of Φ, there exists for each $\mu \in \mathbb{D}$ a uniton factorization of Φ^μ,

$$\Phi^\mu(\lambda, z) = \prod_{j=1}^{m} (\pi_{\alpha_j(\mu,z)} + \lambda \pi^\perp_{\alpha_j(\mu,z)}),$$

which coincides with the original one when $\mu = 1$, and such that each uniton $b_j^\mu(\lambda, z) = \pi_{\alpha_j(\mu,z)} + \lambda \pi^\perp_{\alpha_j(\mu,z)}$ (hence also Φ^μ), is a C^∞-function of $\mu \in \mathbb{D}$, which is real-analytic on any segment through the origin.

THANK YOU!