Titchmarsh-Weyl thoery for vector valued discrete Schrödinger operators

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Later in 1941, Titchmarsh worked on the analytic properties of m functions.

A multidimensional discrete Schrödinger equation is of the form

$$y(n+1) + y(n-1) + B(n)y(n) = zy(n), \ z \in \mathbb{C} \tag{1}$$

where $y(n) = [y_1(n), \ y_2(n), \dots \ y_d(n)]^t$ (t stands for a transpose) and $B(n) \in \mathbb{C}^{d \times d}$.

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where $y(n) = [y_1(n), \;\; y_2(n), \ldots \;\; y_d(n)]^t$ (t stands for a transpose) and $B(n) \in \mathbb{C}^{d \times d}$. This can be generalizes to

$$A(n)y(n+1) + A(n-1)y(n-1) + B(n)y(n) = zy(n), \;\; z \in \mathbb{C} \tag{2}$$

with $A(n), B(n)$ are sequences of $d \times d$ matrices. The equation (2) can be written in the form:

$$\begin{pmatrix} B(1) & A(1) & 0 & \\ A(1) & B(2) & A(2) & \ddots \\ 0 & A(2) & B(3) & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} y(1) \\ y(2) \\ \vdots \\ \vdots \\ \vdots \end{pmatrix} = z \begin{pmatrix} y(1) \\ y(2) \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}.$$

The matrix

$$J = \begin{pmatrix} B(1) & A(1) & 0 & \\ A(1) & B(2) & A(2) & \ddots \\ 0 & A(2) & B(3) & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

is called a block Jacobi matrix.

A transfer matrix $T(m;z) = \begin{pmatrix} z - B(m) & -I \\ I & 0 \end{pmatrix}$ can be used to get a solution. For example:

$$\begin{pmatrix} y(2) \\ y(1) \end{pmatrix} = \begin{pmatrix} z - B(1) & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} y(1) \\ y(0) \end{pmatrix}.$$

Let

$$A(n;z) = T(n;z) \times \cdots \times T(1,z) \times I. \tag{3}$$

The following equation can be used to get a solution at cite m from cite n

$$\begin{pmatrix} y(n+1) \\ y(n) \end{pmatrix} = A(n,m;z) \begin{pmatrix} y(m+1) \\ y(m) \end{pmatrix}$$

Equation (1) induce an operator on $\ell^2(\mathbb{N}, \mathbb{C}^d)$ of the form

$$Jy(n) = y(n+1) + y(n-1) + B(n)y(n).$$

If $B(n) \in \mathbb{C}^{d\times d},\ B(n)^* = B(n)$, then J is a self-adjoint operator on $l^2(\mathbb{N}, \mathbb{C}^d)$.

For $z \in \mathbb{C}$, let

$$\begin{aligned} U(n,z) &= [u_1(n), u_2(n), \ldots, u_d(n)], \\ u_i(n) &= [u_{1,i}(n)\ u_{2,i}(n)\ \ \ldots u_{d,i}(n)]^t \\ V(n,z) &= [v_1(n), v_2(n), \ldots, v_d(n)] \\ v_i(n) &= [v_{1,i}(n)\ v_{2,i}(n)\ \ \ldots v_{d,i}(n)]^t \end{aligned} \tag{4}$$

be the sets of solutions satisfying the following initial conditions

$$U(0,z) = -I, \qquad V(0,z) = 0, \qquad U(1,z) = 0, \qquad V(1,z) = I. \tag{5}$$

Let $z \in \mathbb{C}^+ = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$. The Titchmarsh-Weyl m function is defined as the unique $M(z) \in \mathbb{C}^{d\times d}$ for which

$$F(n,z) = U(n,z) + V(n,z)M(z) \tag{6}$$

where $U(n,z), V(n,z)$ are matrix valued solutions consisting of d linearly independent solutions with initial values (5) and the matrix valued solution $F(n,z)$ is a set of d linearly independent solutions of (1) that are in $l^2(\mathbb{N}, \mathbb{C}^d)$.

Theorem

If F consists of d solutions of (1) in $l^2(\mathbb{N}, \mathbb{C}^d)$. Then

$$M(z) = -F(1,z)F(0,z)^{-1}. \tag{7}$$

Moreover,

$$M(z) = (m_{ij}(z))_{d\times d} \in \mathbb{C}^{d\times d},\ m_{ij}(z) = \langle \delta_j, (J-z)^{-1}\delta_i \rangle. \tag{8}$$

By functional calculus,

$$m_{ij}(z) = \int_{\mathbb{R}} \frac{1}{t-z} d\mu_{ij}$$

where μ_{ij} is a spectral measure for the vectors δ_j and δ_i. Therefore,

$$M(z) = \int_{\mathbb{R}} \frac{1}{t-z} d\mu, \quad \mu = (\mu_{ij})_{d\times d}$$

and

$$M(z) = \int_{\mathbb{R}} \frac{1}{t-z} d\mu = \Big(\int_{\mathbb{R}} \frac{1}{t-z} d\mu_{ij} \Big)_{d\times d}$$

The matrix valued measure μ is a spectral measure of the multidimensional discrete Schrödinger operator J.

$M(z)^* = M(\bar{z})$ and $\operatorname{Im} M(z) = \frac{1}{2i}(M(z) - M(z)^*) > 0$. Hence, for $z \in \mathbb{C}^+$, the map $z \mapsto M(z)$ maps complex upper half plane to Seigel half space.

Titchmarsh-Weyl circles and disks:

Consider the equation (1) on a finite interval [0, N] and

$$F(n,z) = U(n,z) + V(n,z)M_N^{\beta}(z)$$

satisfying a boundary conditions
$\beta_2 F(N,z) + \beta_1 F(N+1,z) = 0$ where

$$\beta = [\beta_1, \beta_2] \in \mathbb{C}^{d\times 2d},\ \beta_1, \beta_2 \in \mathbb{C}^{d\times d}\ \beta^*\beta = I,\ \ \beta J \beta^* = 0. \tag{9}$$

The unique coefficient $M_N^{\beta}(z)$ is called the Weyl m function on the interval $[0, N]$.

$$M_N^\beta(z) = -\big(\beta_2 V(N,z) + \beta_1 V(N+1,z)\big)^{-1}\big(\beta_2 U(N,z) + \beta_1 U(N+1,$$
$$= -\big(\beta_1^{-1}\beta_2 V(N,z) + V(N+1,z)\big)^{-1}\big(\beta_1^{-1}\beta_2 U(N,z) + U(N+$$
$$= -\big(\gamma V(N,z) + V(N+1,z)\big)^{-1}\big(\gamma U(N,z) + U(N+1,z)\big), \;\; \gamma$$

Theorem

The weyl m function $M_N^\beta(z)$ on $[0,N]$ is symmetric and $\operatorname{Im} M_N^\beta(z) > 0$.

Let $\mathcal{W}(N,z,M) = \begin{pmatrix} U(N+1,z) & V(N+1,z) \\ U(N,z) & V(N,z) \end{pmatrix} \begin{bmatrix} I \\ M \end{bmatrix}$. Define a matrix function

$$E(M,N) = -i\mathcal{W}(N,z,M)^* J \mathcal{W}(N,z,M)$$

Observe that

$$\begin{aligned} E(M,N) &= -i[F(N+1,z)^*, F(N,z)^*] J \begin{bmatrix} F(N+1,z) \\ F(N,z) \end{bmatrix} \\ &= -iW_N(\bar{F}, F) \\ &= -2\operatorname{Im} M + 2\operatorname{Im} z \sum_{j=0}^{N} F(j,z)^* F(j,z). \end{aligned} \tag{10}$$

Let $z \in \mathbb{C}^+$. The set

$$\mathcal{D}(N, z) = \{M \in C^{d\times d} | E(M, N) \leq 0\}$$

and

$$C(N, z) = \{M \in C^{d\times d} | E(M, N) = 0\}$$

are respectively called the Weyl disk and Weyl circle.

Nesting Property:

$$\mathcal{D}(N+1,z) \subset \mathcal{D}(N,z), \;\; N \in \mathbb{N}$$

$$\begin{aligned} E(M,N) &= -2\,\mathrm{Im}\, M + 2\,\mathrm{Im}\, z \sum_{j=0}^{N} F(j,z)^* F(j,z) \\ &\leq -2\,\mathrm{Im}\, M + 2\,\mathrm{Im}\, z \sum_{j=0}^{N+1} F(j,z)^* F(j,z) \\ &= E(M,N+1). \end{aligned}$$

$$E(M,N) = -i\Big\{[M - W_N(\bar{V},V)^{-1}W_N(\bar{U},V)^*]^*W_N(\bar{V},V)$$
$$[M - W_N(\bar{V},V)^{-1}W_N(\bar{U},V)^*] - W_N(V,\bar{V})^{-1}\Big\}.$$

Thus it can be expressed as

$$E(M,N) = -[(M - C_N(z))^*R(N,z)^{-2}(M - C_N(z)) - R(N,\bar{z})^2]$$

where $C_N(z) = W_N(\bar{V},V)^{-1}W_N(\bar{U},V)^*$ and $R(N,z) = (iW_N(\bar{V},V))^{-1/2}$. So the equation of Weyl circle can be written as

$$(M - C_N(z))^*R(N,z)^{-2}(M - C_N(z)) = R(N,\bar{z})^2$$

Theorem

For all $z \in \mathbb{C}^+$, $\lim_{N\to\infty} R(N,z)$ *exists and* $\lim_{N\to\infty} R(N,z) \geq 0$. *Also* $\lim_{N\to\infty} C_N(z)$ *exists.*

Green's Identity implies:
$2\,\mathrm{Im}\,z \sum_{j=0}^{N} V(j,z)^* V(j,z) = iW_N(\bar{V}, V) = R(N,z)^{-2} > 0$

Let $C_0(z) = \lim_{N\to\infty} C_N(z)$ and $R_0(z) = \lim_{N\to\infty} R(N,z)$.
Define
$D_0(z) = \{M \in \mathbb{C}^{d\times d} : (M - C_0(z))^* R_0(z)^{-2}(M - C_0(z)) \leq R_0(\bar{z})^2\}$
then

$$D_0(z) = \cap_{N\geq 1} D(N,z).$$

Theorem

Let $z \in \mathbb{C}^+$ *and* $M \in \mathbb{C}^{d\times d}$*. Then for* $F(N,z) = U(N,z) + V(N,z)M$ *we have*
(1) M *is inside* $\mathcal{D}_0(z)$ *if and only if*

$$\sum_{N=1}^{\infty} F(N,z)^* F(N,z) \leq \frac{\operatorname{Im} M}{\operatorname{Im} z}$$

(2) M *is on the boundary of* $\mathcal{D}_0(z)$ *if and only if*

$$\sum_{N=1}^{\infty} F(N,z)^* F(N,z) = \frac{\operatorname{Im} M}{\operatorname{Im} z}$$

Thank you for your attention!