CONGRUENCES FOR BROKEN k-DIAMOND PARTITIONS

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Abstract. We prove two conjectures of Paule and Radu from their recent paper on broken k-diamond partitions.

1. Introduction and Statement of Results

In [1], Paule and Andrews constructed a class of directed graphs called broken k-diamonds, and defined $\Delta_k(n)$ to be the number of broken k-diamond partitions of n. They noted that the generating function for $\Delta_k(n)$ is essentially a modular form. More precisely, if $k \geq 1$, then

$$\sum_{n=0}^{\infty} \Delta_k(n) q^n = q^{(k+1)/12} \frac{\eta(2z)\eta((2k+1)z)}{\eta(z)^3 \eta((4k+2)z)}, \tag{1.1}$$

where $q := e^{2\pi i z}$ and $\eta(z)$ is Dedekind's eta function

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

One can show various congruences for $\Delta_k(n)$ for n in certain arithmetic progressions. For example, Xiong [4] proved congruences for $\Delta_3(n)$ and $\Delta_5(n)$ which had been conjectured by Paule and Radu in [3]. In particular, he showed that

$$\prod_{n=1}^{\infty} (1 - q^n)^4 (1 - q^{2n})^6 \equiv 6 \sum_{n=0}^{\infty} \Delta_3(7n + 5) q^n \pmod 7 \tag{1.2}$$

In this note, we prove the remaining two conjectures in [3]. First, we use (1.2) to prove the following statement (which is denoted Conjecture 3.2 in [3]).

Theorem 1.1. *For all $n \in \mathbb{N}$, we have that*

$$\Delta_3(7^3 n + 82) \equiv \Delta_3(7^3 n + 229) \equiv \Delta_3(7^3 n + 278) \equiv \Delta_3(7^3 n + 327) \equiv 0 \pmod{7}.$$

Now, recall that the weight k Eisenstein series (where $k \geq 4$ is even) are given by

$$E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where B_k is the kth Bernoulli number, and $\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$. Also define

$$\sum_{n=0}^{\infty} c(n) q^n := E_4(2z) \prod_{n=1}^{\infty} (1 - q^n)^8 (1 - q^{2n})^2 = q^{-1/2} E_4(2z) \eta(z)^8 \eta(2z)^2. \tag{1.3}$$

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The coefficients $c(n)$ are of interest here because they are related to broken k-diamond partitions in the following way (as conjectured in [3] and proved in [4]):

$$c(n) \equiv 8\Delta_5(11n+6) \pmod{11}. \tag{1.4}$$

Here we prove the last remaining conjecture of Paule and Radu (which is Conjecture 3.4 of [3]). More precisely, we have the following theorem.

Theorem 1.2. *For every prime $p \equiv 1 \pmod 4$, there exists an integer $y(p)$ such that*

$$c\left(pn+\frac{p-1}{2}\right)+p^8 c\left(\frac{n-(p-1)/2}{p}\right)=y(p)c(n)$$

for all $n \in \mathbb{N}$.

Remark 1. Theorem 1.2 follows from a more technical result (see Theorem 3.1 which is proved in Section 3).

Remark 2. As noted in [3], one can combine (1.4) with Theorem 1.2 to see that for every prime $p \equiv 1 \pmod 4$ and $n \in \mathbb{N}$ we have

$$\Delta_5\left((11n+6)p-\frac{p-1}{2}\right)+p^8\Delta_5\left(\frac{11n+6}{p}+\frac{p-1}{2p}\right)\equiv y(p)\Delta_5(11n+6) \pmod{11}.$$

To prove Theorems 1.1 and 1.2, we make use of the theory of modular forms. In particular, we shall make use of the U-operator, Hecke operators, the theory of twists, and a theorem of Sturm. These results are described in [2]. We shall freely assume standard definitions and notation which may be found there.

2. Proof of Theorem 1.1

First we consider the form $\eta(3z)^4\eta(6z)^6$. By Theorems 1.64 and 1.65 in [2], we have that $\eta(3z)^4\eta(6z)^6 \in S_5\left(\Gamma_0(72), \left(\frac{-1}{\bullet}\right)\right)$. Note from (1.2) that

$$\eta(3z)^4\eta(6z)^6 \equiv 6\sum_{n=0}^{\infty}\Delta_3(7n+5)q^{3n+2} \pmod 7.$$

It follows that

$$f(z) := \eta(3z)^4\eta(6z)^6 \mid U_7 \equiv 6\sum_{n=0}^{\infty}\Delta_3(7^2n+33)q^{3n+2} \pmod 7.$$

Here, U_d denotes Atkin's U-operator, which is defined by

$$\sum_{n=0}^{\infty}a(n)q^n \mid U_d = \sum_{n=0}^{\infty}a(dn)q^n$$

for d a positive integer. By the theory of the U-operator (see Proposition 2.22 and Remark 2.23 in [2]), it follows that $f(z) \in S_5\left(\Gamma_0(504), \left(\frac{-1}{\bullet}\right)\right)$. Now if we define $b(n)$ by

$$\sum_{n=0}^{\infty}b(n)q^n := f(z),$$

then our goal is to show that

$$b(21n+5) \equiv b(21n+14) \equiv b(21n+17) \equiv b(21n+20) \equiv 0 \pmod 7.$$

In order to prove the desired congruence, consider the Dirichlet character ψ defined by $\psi(\bullet) := \left(\frac{\bullet}{7}\right)$. We may consider the ψ-twist of f, which is given by

$$f_\psi(z) := \sum_{n=0}^{\infty} \psi(n) b(n) q^n.$$

By Proposition 2.8 of [2], we have that $f_\psi(z) \in S_5\left(\Gamma_0(24696), \left(\frac{-1}{\bullet}\right)\right)$.

Then consider

$$f(z) - f_\psi(z) = \sum_{n=0}^{\infty} \left(1 - \left(\frac{n}{7}\right)\right) b(n) q^n \in S_5\left(\Gamma_0(24696), \left(\frac{-1}{\bullet}\right)\right).$$

In fact, $f(z) - f_\psi(z) \equiv 0 \pmod 7$. This follows from a theorem of Sturm (see Theorem 2.58 in [2]), which states that $f(z) - f_\psi(z) \equiv 0 \pmod 7$ if its first 23520 coefficients are 0 (mod 7) (which was verified using Maple). Thus we have that

$$\left(1 - \left(\frac{n}{7}\right)\right) b(n) \equiv 0 \pmod 7$$

for all n, and thus

$$b(21n+5) \equiv b(21n+14) \equiv b(21n+17) \equiv b(21n+20) \equiv 0 \pmod 7$$

for all $n \in \mathbb{N}$, as desired.

3. Proof of Theorem 1.2

3.1. **Preliminaries.** Let us first recall the Hecke operators and their properties. If $f(z) = \sum_{n=0}^{\infty} a(n) q^n \in M_k(\Gamma_0(N), \chi)$ and p is prime, the Hecke operator $T_{p,k,\chi}$ (or simply T_p if the weight and character are known from context) is defined by

$$f(z) \mid T_p := \sum_{n=0}^{\infty} \left(a(pn) + \chi(p) p^{k-1} a(n/p)\right) q^n,$$

where we set $a(n/p) = 0$ if $p \nmid n$. It is important to note that $f(z) \mid T_p \in M_k(\Gamma_0(N), \chi)$.

In order to prove the final statement of Theorem 1.2, define

$$g(z) = \sum_{n=0}^{\infty} c_0(n) q^n := E_4(4z)\eta(2z)^8\eta(4z)^2 \in S_9\left(\Gamma_0(16), \left(\frac{-4}{\bullet}\right)\right)$$

and note that $c(n) = c_0(2n+1)$. Thus we wish to show that for every prime $p \equiv 1 \pmod 4$ there exists an integer $y(p)$ such that

$$c_0\left(p(2n+1)\right) + p^8 c_0\left(\frac{2n+1}{p}\right) = y(p) c_0(2n+1)$$

for all $n \in \mathbb{N}$. By summing (and noting that $c_0(n) = 0$ when n is even) we see that this is equivalent to the statement that

$$g(z) \mid T_p = y(p) g(z).$$

That is, we need only show that $g(z)$ is an eigenform of the Hecke operator T_p for all $p \equiv 1 \pmod 4$.

To see this, we let F be the weight 2 Eisenstein series (see (1.18) of [2]) given by

$$F(z) := \frac{\eta(4z)^8}{\eta(2z)^4} = \sum_{n=0}^{\infty} \sigma_1(2n+1) q^{2n+1} \in M_2(\Gamma_0(4)),$$

let $\theta_0(z)$ be the theta-function given by

$$\theta_0(z) := \sum_{n=-\infty}^{\infty} q^{n^2} \in M_{1/2}(\Gamma_0(4)),$$

and let $h(z)$ be the normalized cusp form

$$h(z) := \eta(4z)^6 = \sum_{n=1}^{\infty} a(n) q^n = q - 6q^5 + 9q^9 + \cdots \in S_3\left(\Gamma_0(16), \left(\frac{-4}{\bullet}\right)\right).$$

Then $h(z)$ is a modular form with complex multiplication, and for primes p we have (see Section 1.2.2 of [2])

$$a(p) = \begin{cases} 2x^2 - 2y^2 & p = x^2 + y^2 \text{ with } x, y \in \mathbb{Z} \text{ and } x \text{ odd} \\ 0 & p \equiv 2, 3 \pmod 4. \end{cases}$$

Then we may define $f_1, f_2, f \in S_9\left(\Gamma_0(16), \left(\frac{-4}{\bullet}\right)\right)$ by

$$f_1(z) = \sum_{n=0}^{\infty} d_1(n) q^n := E_4(4z) F(z) \left[4\theta_0^6(4z) - \theta_0^6(2z) + 4\theta_0^4(2z)\theta_0^2(4z) - 6\theta_0^2(2z)\theta_0^4(4z)\right]$$

$$f_2(z) = \sum_{n=0}^{\infty} d_2(n) q^n := E_4(4z) F(2z) h(z)$$

$$f(z) = \sum_{n=0}^{\infty} d(n) q^n := f_1(z) + 8i\sqrt{3} f_2(z).$$

We prove the following theorem involving these forms.

Theorem 3.1. *The forms $f(z)$ and $\overline{f(z)}$ are eigenforms of the Hecke operator T_p for all primes p. Furthermore we have that*

$$\mathbb{T}_g = \langle f, \overline{f} \rangle,$$

where $\mathbb{T}_g$ is the subspace of $S_9\left(\Gamma_0(16), \left(\frac{-4}{\bullet}\right)\right)$ spanned by g together with $g \mid T_p$ for all primes p.

Proof. First note that f and $\overline{f}$ are eigenforms of the Hecke operator T_p for all primes p. To see this, note that there is a basis of Hecke eigenforms of the space $S_9\left(\Gamma_0(16), \left(\frac{-4}{\bullet}\right)\right)$. Also, both f and $\overline{f}$ are eigenforms of T_5 with eigenvalue 258, and one can compute that this eigenspace

$$\ker(T_5 - 258)$$

is 2-dimensional (this can be done, for example, by computing the characteristic polynomial of T_5 using Sage). Finally, both f and $\overline{f}$ are eigenforms of the Hecke operator T_7, and they have different eigenvalues.

Now, note that

$$g = \left(\frac{1}{2} + \frac{i}{2\sqrt{3}}\right) f + \left(\frac{1}{2} - \frac{i}{2\sqrt{3}}\right) \overline{f}$$

and thus $\mathbb{T}_g$ is a two-dimensional subspace of $\langle f, \overline{f}\rangle$. Thus $\mathbb{T}_g = \langle f, \overline{f}\rangle$, as desired. $\square$

3.2. **Proof of Theorem 1.2.** Suppose p is a prime with $p \equiv 1 \pmod 4$. Then we need only check that f and $\overline{f}$ are eigenforms of T_p with the *same eigenvalue*. Since these eigenvalues are the coefficients of q^p in the expansions of f and $\overline{f}$ (see Proposition 2.6 of [2]), we need only show that

$$d(p) = \overline{d(p)},$$

i.e., $d(p) \in \mathbb{R}$.

Now, note that the coefficients of $E_4(4z)$ are only supported on indices that are congruent to 0 mod 4 by construction. Also, the coefficients of $F(2z)$ are supported on indices which are 2 (mod 4), and the coefficients of $h(z)$ are supported on indices which are 1 (mod 4). Thus the coefficients of f_2 are only supported on indices that are congruent to 3 mod 4, so we have that $d_2(p) = 0$, and thus $d(p) = d_1(p) \in \mathbb{R}$, as desired.

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