A REFINEMENT OF RAMANUJAN'S CONGRUENCES MODULO POWERS OF 7 AND 11

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Abstract. Ramanujan's famous congruences for the partition function modulo powers of 5, 7, and 11 have inspired much further research. For example, in 2002 Lovejoy and Ono found subprogressions of $5^j n + \beta_5(j)$ for which Ramanujan's congruence mod 5^j could be strengthened to a statement modulo 5^{j+1}. Here we provide the analogous results modulo powers of 7 and 11. We require the arithmetic properties of two special elliptic curves.

1. Introduction and Statement of Results

A *partition* of a positive integer n is a non-increasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of partitions of n (where by convention we set $p(0) = 1$). The famous Ramanujan congruences for $p(n)$ modulo powers of $5, 7,$ and 11 assert that

$$\begin{aligned} p(5^j n + \beta_5(j)) &\equiv 0 \pmod{5^j}, \\ p(7^j n + \beta_7(j)) &\equiv 0 \pmod{7^{\lfloor j/2 \rfloor + 1}}, \\ p(11^j n + \beta_{11}(j)) &\equiv 0 \pmod{11^j} \end{aligned}$$

for all non-negative integers n, where $\beta_m(j) := 1/24 \pmod{m^j}$. They were proved by Ramanujan [2], Watson [11], and Atkin [1].

One of the natural questions stemming from the Ramanujan congruences is whether these results are optimal. In this direction, Lovejoy and Ono [4] defined rational numbers $\beta_5(j, \ell)$ by

$$\beta_5(j,\ell) = \begin{cases} \frac{19 \cdot 5^j \cdot \ell^2 + 1}{24} & \text{if } j \text{ is odd,} \\ \frac{23 \cdot 5^j \cdot \ell^2 + 1}{24} & \text{if } j \text{ is even,} \end{cases}$$

and proved that for a prime $\ell \geq 7$, and a non-negative integer n, we have that

$$\begin{aligned} p(5^j \ell^2 n + \beta_5(j,\ell)) \equiv \left(\frac{15}{\ell}\right) \left(1 + \ell - \ell^2 \left(\frac{-24n - 19}{\ell}\right)\right) p(5^j n + \beta_5(j)) \\ - \ell p\left(\frac{5^j n}{\ell^2} + \beta_5(j, \ell^{-1})\right) \pmod{5^{j+1}} \end{aligned}$$

if $j \geq 1$ is odd, and

$$p(5^j\ell^2 n + \beta_5(j,\ell)) \equiv \left(\frac{15}{\ell}\right)\left(1+\ell-\left(\frac{-24n-23}{\ell}\right)\right)p(5^j n + \beta_5(j)) \\ - \ell p\left(\frac{5^j n}{\ell^2} + \beta_5(j,\ell^{-1})\right) \pmod{5^{j+1}}$$

if $j \geq 2$ is even. This gives subprogressions of $5^j n + \beta_5(j)$ for which Ramanujan's congruence mod 5^j could be strengthened to a statement modulo 5^{j+1} which was unknown to Ramanujan. To do this, Lovejoy and Ono exploited connections between the generating function for the numbers $p(5^j n + \beta_5(j))$ and certain half integer weight Hecke eigenforms. In this context, their result is equivalent to determining the eigenvalues of the half integer weight Hecke operator $T(\ell^2) \pmod 5$.

In the case of the Ramanujan congruences modulo powers of 7 and 11, the eigenvalues of the corresponding half integer weight Hecke eigenforms are harder to determine modulo 7 and 11. However, these eigenvalues can be described modulo 7 and 11 using the elliptic curves

$$E_7 : y^2 + xy + y = x^3 + x^2 - 4x + 5$$

and

$$E_{11} : y^2 + xy + y = x^3 - 14x + 20.$$

If ℓ is prime, then we let $\#E_7(\mathbb{F}_\ell)$ (resp. $\#E_{11}(\mathbb{F}_\ell)$) denote the number of points on E_7 (resp. E_{11}) over $\mathbb{F}_\ell$, including the point at infinity.

This leads to the following analogous results modulo powers of 7 and 11. Let

$$\beta_7(j,\ell) = \begin{cases} \frac{17\cdot 7^j\cdot\ell^2+1}{24} & \text{if } j \text{ is odd,} \\ \frac{23\cdot 7^j\cdot\ell^2+1}{24} & \text{if } j \text{ is even,} \end{cases} \quad \text{and} \quad \beta_{11}(j,\ell) = \begin{cases} \frac{13\cdot 11^j\cdot\ell^2+1}{24} & \text{if } j \text{ is odd,} \\ \frac{23\cdot 11^j\cdot\ell^2+1}{24} & \text{if } j \text{ is even.} \end{cases} \tag{1.1}$$

Theorem 1.1. *Let $\ell \geq 5$ be prime.*

(1) *If $j \geq 1$ is odd, then for every non-negative integer n we have*

$$p(7^j\ell^2 n + \beta_7(j,\ell)) \equiv \ell\left(\frac{3}{\ell}\right)\left(1+\ell-\#E_7(\mathbb{F}_\ell)-\left(\frac{24n+17}{\ell}\right)\right)p(7^j n + \beta_7(j)) \\ - \ell^3 p\left(\frac{7^j n}{\ell^2} + \beta_7(j,\ell^{-1})\right) \pmod{7^{\lfloor j/2\rfloor+2}}.$$

If $j \geq 2$ is even, then for every non-negative integer n we have

$$p(7^j\ell^2 n + \beta_7(j,\ell)) \equiv \ell\left(\frac{3}{\ell}\right)\left(1+\ell-\#E_7(\mathbb{F}_\ell)-\ell^3\left(\frac{-24n-23}{\ell}\right)\right)p(7^j n + \beta_7(j)) \\ - \ell^3 p\left(\frac{7^j n}{\ell^2} + \beta_7(j,\ell^{-1})\right) \pmod{7^{\lfloor j/2\rfloor+2}}.$$

(2) *If $j \geq 1$ is odd, then for every non-negative integer n we have*

$$p(11^j\ell^2 n+\beta_{11}(j,\ell))\equiv \ell^3\left(\frac{3}{\ell}\right)\left(1+\ell-\#E_{11}(\mathbb{F}_\ell)-\left(\frac{24n+13}{\ell}\right)\right)p(11^jn+\beta_{11}(j))$$
$$-\ell^7 p\left(\frac{11^jn}{\ell^2}+\beta_{11}(j,\ell^{-1})\right)\pmod{11^{j+1}}.$$

If $j\geq 2$ is even, then for every non-negative integer n we have

$$p(11^j\ell^2 n+\beta_{11}(j,\ell))\equiv \ell^3\left(\frac{3}{\ell}\right)\left(1+\ell-\#E_{11}(\mathbb{F}_\ell)-\ell^5\left(\frac{-24n-23}{\ell}\right)\right)p(11^jn+\beta_{11}(j))$$
$$-\ell^7 p\left(\frac{11^jn}{\ell^2}+\beta_{11}(j,\ell^{-1})\right)\pmod{11^{j+1}}.$$

As in [4], we have the following corollaries.

Corollary 1.2. *Let $\ell\geq 5$ be prime.*

(1) *Suppose that*
$$\#E_7(\mathbb{F}_\ell)\equiv 1+\ell \pmod 7.$$
If $j\geq 1$ is odd (resp. even), let $0\leq r,s\leq \ell-1$ be integers such that
 (a) $24r+17\equiv 0 \pmod{\ell}$ *(resp.* $24r+23\equiv 0 \pmod{\ell}$*)*
 (b) $24(r+s\ell)+17\not\equiv 0 \pmod{\ell^2}$ *(resp.* $24(r+s\ell)+23\not\equiv 0 \pmod{\ell^2}$*).*
Then for every non-negative integer N we have that
$$p(7^j\ell^2(\ell^2N+\ell s+r)+\beta_7(j,\ell))\equiv 0 \pmod{7^{\lfloor j/2\rfloor+2}}.$$

(2) *Suppose that*
$$\#E_{11}(\mathbb{F}_\ell)\equiv 1+\ell \pmod{11}.$$
If $j\geq 1$ is odd (resp. even), let $0\leq r,s\leq \ell-1$ be integers such that
 (a) $24r+13\equiv 0 \pmod{\ell}$ *(resp.* $24r+23\equiv 0 \pmod{\ell}$*)*
 (b) $24(r+s\ell)+13\not\equiv 0 \pmod{\ell^2}$ *(resp.* $24(r+s\ell)+23\not\equiv 0 \pmod{\ell^2}$*).*
Then for every non-negative integer N we have that
$$p(11^j\ell^2(\ell^2N+\ell s+r)+\beta_{11}(j,\ell))\equiv 0 \pmod{11^{j+1}}.$$

Furthermore, the proportion of primes ℓ for which $\#E_7(\mathbb{F}_\ell)\equiv 1+\ell \pmod 7$ is $\frac{7}{48}$, and the proportion of primes ℓ for which $\#E_{11}(\mathbb{F}_\ell)\equiv 1+\ell \pmod{11}$ is $\frac{11}{120}$.

Corollary 1.3. *Let $\ell\geq 5$ be prime.*

(1) *Suppose that*
$$\#E_7(\mathbb{F}_\ell)\equiv 1+\ell-i \pmod 7,$$
where $i=\pm1$. If $j\geq 1$ is odd (resp. even), let $0\leq r\leq \ell-1$ be an integer such that
$$\left(\frac{24r+17}{\ell}\right)\equiv i \pmod 7 \qquad \left(\textit{resp.}\ \left(\frac{-24r-23}{\ell}\right)\equiv \ell^3 i \pmod 7\right).$$
Then for every non-negative integer N we have that
$$p(7^j\ell^2(\ell N+r)+\beta_7(j,\ell))\equiv 0 \pmod{7^{\lfloor j/2\rfloor+2}}.$$

(2) *Suppose that*

$$\#E_{11}(\mathbb{F}_\ell) \equiv 1+\ell-i \pmod{11},$$

where $i=\pm 1$. *If* $j \geq 1$ *is odd (resp. even), let* $0 \leq r \leq \ell-1$ *be an integer such that*

$$\left(\frac{24r+13}{\ell}\right) \equiv i \pmod{11} \qquad \left(\textit{resp. } \left(\frac{-24r-23}{\ell}\right) \equiv \ell^5 i \pmod{11}\right).$$

Then for every non-negative integer N *we have that*

$$p(11^j\ell^2(\ell N+r)+\beta_{11}(j,\ell)) \equiv 0 \pmod{11^{j+1}}.$$

Furthermore, the proportion of primes ℓ *for which* $\#E_7(\mathbb{F}_\ell) \equiv 1+\ell-i \pmod 7$ *is* $\frac{41}{288}$, *and the proportion of primes* ℓ *for which* $\#E_{11}(\mathbb{F}_\ell) \equiv 1+\ell-i \pmod{11}$ *is* $\frac{109}{1200}$.

Remark 1. Note that the only valid values of i are ± 1 since $\ell^3 \equiv \pm 1 \pmod 7$ and $\ell^5 \equiv \pm 1 \pmod{11}$ for all ℓ.

For convenience, we will give a few values of ℓ for which these corollaries will apply.

p	i	ℓ such that $\#E_p(\mathbb{F}_\ell) \equiv 1+\ell-i \pmod p$
7	0	31, 47, 79, 97, 113, 127, 191, ...
7	-1	7, 13 ,53, 61, 149, 151, 163, 167, ...
7	1	23, 41, 71, 89, 103, 131, 173, 199, ...
11	0	11, 19, 23, 41, 59, 107, 193, ...
11	-1	5, 101, 137, 167, 181, ...
11	1	67, 109, 131, 139, 149, 179, ...

In addition, we give numerical evidence for the proven proportions by considering the primes up to 30,000. Let $\pi(X;p,i) := \#\{\text{primes } \ell \leq X : \#E_p(\mathbb{F}_\ell) \equiv 1+\ell-i \pmod p\}$ and let $\pi(X) := \{\text{primes } \ell \leq X\}$.

p	i	$\frac{\pi(30000;p,i)}{\pi(30000)}$	$\lim_{X\to\infty} \frac{\pi(X;p,i)}{\pi(X)}$
7	0	$0.1433\ldots$	$0.1458\ldots$
7	1	$0.1458\ldots$	$0.1423\ldots$
7	-1	$0.1529\ldots$	$0.1423\ldots$
11	0	$0.0860\ldots$	$0.0916\ldots$
11	1	$0.0878\ldots$	$0.0908\ldots$
11	-1	$0.0924\ldots$	$0.0908\ldots$

In Section 2, we will describe the connection between the theory of modular forms and the generating functions for $p(n)$ in the arithmetic progressions of interest. In Section 3 we will see that the half integer weight modular forms which arise are Hecke eigenforms, and work to understand their eigenvalues modulo 7 and 11 using the elliptic curves E_7 and E_{11}. In Section 4 we will use this knowledge to prove Theorem 1.1 and Corollaries 1.2 and 1.3.

2. The Work of Watson and Atkin

In this section, we will reconsider the work of Watson and Atkin in order to establish the connection between half integer weight modular forms and the generating functions for $p(7^jn+\beta_7(j))$ and $p(11^jn+\beta_{11}(j))$. First, let us consider the generating function for $p(7^jn+\beta_7(j))$.

Theorem 2.1 (page 124 of [11])**.** *If $j \geq 1$, then the generating function for the numbers $p(7^j n + \beta_7(j))$ is of the form*

$$\sum_{n=0}^{\infty} p(7^j n + \beta_7(j))q^n = \begin{cases} \sum_{i\geq 1} \left(x_{j,i} q^{i-1} \prod_{n=1}^{\infty} \frac{(1-q^{7n})^{4i-1}}{(1-q^n)^{4i}} \right) & \text{if } j \text{ is odd,} \\ \sum_{i\geq 1} \left(x_{j,i} q^{i-1} \prod_{n=1}^{\infty} \frac{(1-q^{7n})^{4i}}{(1-q^n)^{4i+1}} \right) & \text{if } j \text{ is even,} \end{cases}$$

where

$$x_{j,i} \equiv \begin{cases} 5^{\lfloor j/2 \rfloor} 7^{\lfloor j/2 \rfloor + 1} \pmod{7^{\lfloor j/2 \rfloor + 2}} & \text{if } i = 1, \\ 0 \pmod{7^{\lfloor j/2 \rfloor + 2}} & \text{otherwise.} \end{cases}$$

In order to connect this result to the theory of modular forms, we first recall Dedekind's eta-function, which is given by

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n),$$

where $q := e^{2\pi i z}$. If χ_{12} is the quadratic character given by the Jacobi symbol $\chi_{12}(n) := \left(\frac{12}{n}\right)$, then we define

$$F_7(z) := \eta^{17}(24z) = \sum_{n=0}^{\infty} a_7(n) q^n$$

$$G_7(z) := \eta^{23}(24z) = \sum_{n=0}^{\infty} b_7(n) q^n$$

and note that (see [6]) $F_7(z) \in S_{17/2}(\Gamma_0(576), \chi_{12})$ and $G_7(z) \in S_{23/2}(\Gamma_0(576), \chi_{12})$. Then Theorem 2.1 immediately implies the following statement.

Corollary 2.2. *If $j \geq 1$, then for every non-negative integer n we have*

$$p(7^j n + \beta_7(j)) \equiv \begin{cases} 5^{\lfloor j/2 \rfloor} 7^{\lfloor j/2 \rfloor + 1} a_7(24n + 17) \pmod{7^{\lfloor j/2 \rfloor + 2}} & \text{if } j \text{ is odd,} \\ 5^{\lfloor j/2 \rfloor} 7^{\lfloor j/2 \rfloor + 1} b_7(24n + 23) \pmod{7^{\lfloor j/2 \rfloor + 2}} & \text{if } j \text{ is even.} \end{cases}$$

Proof. As in [4], note that when j is odd, we have

$$\frac{1}{5^{\lfloor j/2 \rfloor} 7^{\lfloor j/2 \rfloor + 1}} \sum_{n=1}^{\infty} p(7^j n + \beta_7(j)) q^{24n+17} \equiv q^{17} \prod_{n=1}^{\infty} \frac{(1 - q^{7 \cdot 24n})^3}{(1 - q^{24n})^4} \equiv \sum_{n=1}^{\infty} a_7(n) q^n \pmod{7},$$

as desired. Similarly, when j is even, we have

$$\frac{1}{5^{\lfloor j/2 \rfloor} 7^{\lfloor j/2 \rfloor + 1}} \sum_{n=1}^{\infty} p(7^j n + \beta_7(j)) q^{24n+23} \equiv q^{23} \prod_{n=1}^{\infty} \frac{(1 - q^{7 \cdot 24n})^4}{(1 - q^{24n})^5} \equiv \sum_{n=1}^{\infty} b_7(n) q^n \pmod{7},$$

as desired. □

In order to understand the generating function of $p(11^j n + \beta_{11}(j))$, we first recall that the classical Eisenstein series $E_k(z)$ (for even $k \geq 2$) is given by

$$E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where B_k is the kth Bernoulli number and $\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$. Set

$$F_{11}(z) := \eta^{13}(24z)E_8(24z) = \sum_{n=0}^{\infty} a_{11}(n)q^n$$

$$G_{11}(z) := \eta^{23}(24z)E_8(24z) = \sum_{n=0}^{\infty} b_{11}(n)q^n.$$

Then it follows from the work of Atkin [1] that

Corollary 2.3. *If $j \geq 1$, then for every non-negative integer n we have*

$$p(11^j n + \beta_{11}(j)) \equiv \begin{cases} 4^{3(j-1)/2}11^j a_{11}(24n+13) \pmod{11^{j+1}} & \text{if } j \text{ is odd,} \\ 4^{(3j-4)/2}11^j b_{11}(24n+23) \pmod{11^{j+1}} & \text{if } j \text{ is even.} \end{cases}$$

Proof. In [1], Atkin proved the Ramanujan congruences modulo powers of 11 by defining a sequence of functions $L_j(z)$ by $L_1(z) := \phi(z) \mid U_{11}$ and

$$L_{2j}(z) := L_{2j-1}(z) \mid U_{11}$$
$$L_{2j+1}(z) := (\phi(z)L_{2j}(z)) \mid U_{11}$$

for $j \geq 1$, where U_{11} is the usual U-operator and $\phi(z) := \frac{\eta(121z)}{\eta(z)}$. He proved that

$$L_j(z) = \begin{cases} q^{13/24}\eta(11z)\sum_{n=0}^{\infty} p(11^j n + \beta_{11}(j))q^n & j \text{ odd} \\ q^{23/24}\eta(z)\sum_{n=0}^{\infty} p(11^j n + \beta_{11}(j))q^n & j \text{ even,} \end{cases}$$

and achieved his result by showing that $11^{-j}L_j(z)$ has integral coefficients for all $j \geq 1$. In order to do this, he defined a basis of modular functions on $\Gamma_0(11)$ which includes functions $g_n(z)$ for $n \geq 2$ (defined explicitly in Appendix A of [1]). He proved that there exist integers $c_{j,r}$ such that

$$11^{-j}L_j(z) = \sum_r c_{j,r}g_n(z)$$

where $c_{j,3} \equiv c_{j,4} \equiv 0 \pmod{11}$ for all j, $c_{j,4} \equiv 0 \pmod{11^2}$ when j is even, and $c_{j,r} \equiv 0 \pmod{11^2}$ for all j and $r \geq 5$. Thus we have

$$11^{-j}L_{j+1} \equiv \begin{cases} c_{j,2}g_2(z) \mid U_{11} + c_{j,3}g_3(z) \mid U_{11} + c_{j,4}g_4(z) \mid U_{11} \pmod{11^2} & j \text{ odd} \\ c_{j,2}\left(\phi(z)g_2(z)\right) \mid U_{11} + c_{j,3}\left(\phi(z)g_3(z)\right) \mid U_{11} \pmod{11^2} & j \text{ even.} \end{cases}$$

Now, one can compute that $g_2(z) \mid U_{11} \equiv 4 \cdot 11 g_2(z) \pmod{11^2}$, $(\phi(z)g_2(z)) \mid U_{11} \equiv 5 \cdot 11 g_2(z) \pmod{11^2}$, and $g_3(z) \mid U_{11} \equiv g_4(z) \mid U_{11} \equiv (\phi(z)g_3(z)) \mid U_{11} \equiv 0 \pmod{11}$. Thus we have that

$$11^{-(j+1)}L_{j+1} \equiv \begin{cases} 4c_{j,2}g_2(z) \pmod{11} & j \text{ odd} \\ 5c_{j,2}g_2(z) \pmod{11} & j \text{ even.} \end{cases}$$

Then the result follows inductively from the fact that $c_{j,2} = 1$, since (see [1])

$$11^{-1}L_1(z) = g_2(z) + 2 \cdot 11 g_3(z) + 11^2 g_4(z) + 11^3 g_5(z),$$

and the result for $j = 1$, (see Lemma 3.1 of [12]) i.e., that we have

$$g_2(z) \equiv 11^{-1}L_1(z) \equiv \eta(11z)\eta^{13}(z)E_8(z) \pmod{11}.$$

□

3. Properties of the Forms F_i and G_i

We begin by recalling the half integer weight Hecke operators. If $g(z) = \sum_{n=0}^{\infty} \alpha(n)q^n \in M_{\lambda+1/2}(\Gamma_0(4N), \psi)$ is a half integer weight modular form and $p \nmid 4N$ is prime, then the Hecke operator $T_\lambda(p^2)$ is given by

$$g \mid T_\lambda(p^2) := \sum_{n=0}^{\infty} \left(\alpha(p^2 n) + \psi(p) \left(\frac{(-1)^\lambda n}{n} \right) p^{\lambda-1} \alpha(n) + \psi(p^2) p^{2\lambda-1} \alpha(n/p^2) \right) q^n.$$

We say that g is a *Hecke eigenform* if for each prime $p \nmid 4N$ there is a complex number $\lambda_g(p)$ such that

$$g \mid T_\lambda(p^2) = \lambda_p(g)g.$$

Luckily, it is already known that F_i and G_i are Hecke eigenforms (the following are special cases of more general theorems).

Lemma 3.1 (Newman [5]). *If $\ell \geq 5$ is prime, then define $\lambda_{7,a}(\ell)$ and $\lambda_{7,b}(\ell)$ by*

$$\begin{aligned} \lambda_{7,a}(\ell) =& a_7(17\ell^2) + \left(\frac{51}{\ell} \right) \ell^7 \\ \lambda_{7,b}(\ell) =& b_7(23\ell^2) + \left(\frac{-69}{\ell} \right) \ell^{10}. \end{aligned}$$

Then F_7 and G_7 are Hecke eigenforms with eigenvalues given by $\lambda_{7,a}(\ell)$ and $\lambda_{7,b}(\ell)$, respectively. That is, for every positive integer n, we have

$$\begin{aligned} \lambda_{7,a}(\ell) a_7(n) =& a_7(\ell^2 n) + \left(\frac{3n}{\ell} \right) \ell^7 a_7(n) + \ell^{15} a_7(n/\ell^2) \\ \lambda_{7,b}(\ell) b_7(n) =& b_7(\ell^2 n) + \left(\frac{-3n}{\ell} \right) \ell^{10} b_7(n) + \ell^{21} b_7(n/\ell^2). \end{aligned}$$

Lemma 3.2 (Garvan, Cor 3.2 [3]). *If $\ell \geq 5$ is prime, then define $\lambda_{11,a}(\ell)$ and $\lambda_{11,b}(\ell)$ by*

$$\begin{aligned} \lambda_{11,a}(\ell) =& a_{11}(13\ell^2) + \left(\frac{39}{\ell} \right) \ell^{13} \\ \lambda_{11,b}(\ell) =& b_{11}(23\ell^2) + \left(\frac{-69}{\ell} \right) \ell^{18}. \end{aligned}$$

Then F_{11} and G_{11} are Hecke eigenforms with eigenvalues given by $\lambda_{11,a}(\ell)$ and $\lambda_{11,b}(\ell)$, respectively. That is, for every positive integer n, we have

$$\begin{aligned} \lambda_{11,a}(\ell) a_{11}(n) =& a_{11}(\ell^2 n) + \left(\frac{3n}{\ell} \right) \ell^{13} a_{11}(n) + \ell^{27} a_{11}(n/\ell^2) \\ \lambda_{11,b}(\ell) b_{11}(n) =& b_{11}(\ell^2 n) + \left(\frac{-3n}{\ell} \right) \ell^{18} b_{11}(n) + \ell^{37} b_{11}(n/\ell^2). \end{aligned}$$

Remark 2. Although Garvan did not give these formulae for the eigenvalues, they are easily derived using the definition of the Hecke operator given above.

In order to establish Theorem 1.1, the remaining task is to determine the eigenvalues $\lambda_{7,a}(\ell)$ and $\lambda_{7,b}(\ell) \pmod 7$ and $\lambda_{11,a}(\ell)$ and $\lambda_{11,b}(\ell) \pmod{11}$. In order to do this, we will use the Shimura correspondence $S_{t,\lambda}$ to map the half-integer weight eigenforms F_p, G_p to integer weight eigenforms.

Suppose that $g(z) = \sum_{n=1}^{\infty} \alpha(n)q^n \in S_{\lambda+1/2}(\Gamma_0(4N), \psi)$ is an eigenform with $\lambda \geq 2$. Then if t is any squarefree integer, define $A_t(n)$ by

$$\sum_{n=1}^{\infty} \frac{A_t(n)}{n^s} := L(s-\lambda+1, \psi\chi_{-1}^{\lambda}\chi_t) \cdot \sum_{n=1}^{\infty} \frac{\alpha(tn^2)}{n^s},$$

where $\chi_{-1} = \left(\frac{-1}{\bullet}\right)$ and $\chi_t = \left(\frac{t}{\bullet}\right)$. Then these values $A_t(n)$ define the cusp form

$$S_{t,\lambda}(g(z)) := \sum_{n=1}^{\infty} A_t(n)q^n \in S_{2\lambda}(\Gamma_0(2N), \psi^2).$$

It is important to note that the Shimura correspondence $S_{t,\lambda}$ commutes with the Hecke operators of integral and half integral weight, i.e.,

$$S_{t,\lambda}(g \mid T_\lambda(p^2)) = S_{t,\lambda}(g) \mid T_p^{\lambda},$$

where $T_\lambda(p^2)$ and T_p^λ are the usual Hecke operators on $S_{\lambda+1/2}(\Gamma_0(4N), \psi)$ and $S_{2\lambda}(\Gamma_0(2N), \psi^2)$.

Using the theory of modular forms mod ℓ, and the Shimura correspondence, we prove the following:

Theorem 3.3. *If $\ell \geq 5$ is prime, then*

$$\lambda_{7,a}(\ell) \equiv \lambda_{7,b}(\ell) \equiv \ell\left(\frac{3}{\ell}\right)(1+\ell-\#E_7(\mathbb{F}_\ell)) \pmod 7$$
$$\lambda_{11,a}(\ell) \equiv \lambda_{11,b}(\ell) \equiv \ell^3\left(\frac{3}{\ell}\right)(1+\ell-\#E_{11}(\mathbb{F}_\ell)) \pmod{11}.$$

Proof. First recall that $F_7(z) \in S_{17/2}(\Gamma_0(576), \chi_{12})$ and $G_7(z) \in S_{23/2}(\Gamma_0(576), \chi_{12})$ are eigenforms of the half integer weight Hecke operators with eigenvalues $\lambda_{7,a}(\ell)$ and $\lambda_{7,b}(\ell)$.

Now let $\mathfrak{F}_7(z)$ be the image of F_7 under the Shimura correspondence $S_{17,8}$ and $\mathfrak{G}_7(z)$ be the image of G_7 under the Shimura correspondence $S_{23,11}$. Then

$$\begin{aligned}\mathfrak{F}_7(z) &= q + 114810q^5 + \cdots \in S_{16}(\Gamma_0(288), \chi_{\text{triv}})\\ \mathfrak{G}_7(z) &= q + 23245050q^5 + \cdots \in S_{22}(\Gamma_0(288), \chi_{\text{triv}}).\end{aligned}$$

Furthermore, let $H_7(z) = \sum_{n=1}^{\infty} c_7(n)q^n \in S_2(\Gamma_0(42))$ be the cusp form associated to the elliptic curve E_7 (which has conductor 42), and recall that $c_7(\ell) = 1+\ell-\#E_7(\mathbb{F}_\ell)$ for primes $\ell \nmid 42$. Then it suffices to show that

$$\mathfrak{F}_7(z) \equiv \mathfrak{G}_7(z) \equiv \sum_{n=1}^{\infty} \chi_{12}(n)nc_7(n)q^n \pmod 7.$$

In order to see this, first recall Ramanujan's Theta-operator, which is given by

$$\Theta\left(\sum_{n=1}^{\infty} \alpha(n)q^n\right) := \sum_{n=1}^{\infty} n\alpha(n)q^n.$$

It can be shown (see [10], [6]) that

$$\tilde{H} := 12\Theta H_7 - 2H_7E_2$$

is a cusp form of weight 4 and level 42. Now, since we have that $1 \equiv E_6(z)$ and $E_2(z) \equiv E_8(z) \pmod 7$, it follows that

$$E_6 \cdot \Theta H_7 = \frac{E_6}{12}\left(\tilde{H} + 2H_7E_2\right) \equiv \frac{E_6}{12}\left(E_6\tilde{H} + 2H_7E_8\right) \pmod 7,$$

and thus ΘH_7 is congruent (mod 7) to a modular form in $S_{16}(\Gamma_0(42))$. In fact, more is true; it is congruent to a modular form in $S_{16}(\Gamma_0(6))$. This follows from a theorem of Sturm (see [9]) by computing the first few coefficients (in this case, 128) of ΘH_7 and the basis elements of $S_{16}(\Gamma_0(6))$ (the first 200 terms of the q-expansions of the 13 basis elements of $S_{16}(\Gamma_0(6))$ were calculated using Sage). Now, the χ_{12} quadratic twist (see [6])

$$(\Theta H_7)_{\chi_{12}} = \sum_{n=1}^{\infty} n\left(\frac{12}{n}\right)c_7(n)q^n$$

is congruent to a modular form in $S_{16}(\Gamma_0(864))$.

Since $\mathfrak{F}_7(z) \in S_{16}(\Gamma_0(864))$ as well, we may check that

$$\sum_{n=1}^{\infty} n\left(\frac{12}{n}\right)c_7(n)q^n \equiv \mathfrak{F}_7(z) \pmod 7$$

using the theorem of Sturm by checking that the first 2304 coefficients agree (mod 7). However, since H_7 and $\mathfrak{F}_7$ are both eigenforms, it follows that we need only check the coefficients of prime index, i.e., that

$$\ell\left(\frac{12}{\ell}\right)c_7(\ell) \equiv \frac{1}{7}p\left(7 \cdot \frac{17\ell^2 - 17}{24} + \frac{17 \cdot 7 + 1}{24}\right) + \left(\frac{51}{\ell}\right)\ell^7 \pmod 7$$

for all primes $3 < \ell < 2304$ (using the definition of $\lambda_{7,a}(\ell)$ in Lemma 3.1, and Corollary 2.2 in the case where $j = 1$). Although this requires us to compute $p(n)$ for a few values of n up to 26,161,203, it is a fairly short calculation in Sage using the command `Partitions(n).cardinality()` for the right hand side. Alternately, one could compute coefficients of $F_7(z)$ using its definition, noting that $\eta(z)^{17} \equiv \eta(7z)^2\eta(z)^3 \pmod 7$.

Now, to show that $\mathfrak{G}_7 \equiv \sum_{n=1}^{\infty} \chi_{12}(n)nc_7(n)q^n \pmod 7$, we first note that both forms are congruent to modular forms in $S_{22}(\Gamma_0(288))$. Thus by the theorem of Sturm, we need only check the first 1056 coefficients. As before, we need only check that

$$\ell\left(\frac{12}{\ell}\right)c_7(\ell) \equiv \frac{1}{5 \cdot 7^2}p\left(7^2 \cdot \frac{23\ell^2 - 23}{24} + \frac{23 \cdot 7^2 + 1}{24}\right) + \left(\frac{-69}{\ell}\right)\ell^{10} \pmod 7$$

for all primes $3 < \ell < 1056$. This proves the first statement of the theorem.

The second statement is proved similarly: as before, let $\mathfrak{F}_{11}(z)$ be the image of F_{11} under the Shimura correspondence $S_{13,14}$ and $\mathfrak{G}_7(z)$ be the image of G_7 under the Shimura correspondence $S_{23,19}$. Then

$$\mathfrak{F}_{11}(z) = q - 1992850350q^5 + \cdots \in S_{28}(\Gamma_0(288), \chi_{\text{triv}})$$
$$\mathfrak{G}_{11}(z) = q - 4477461318150q^5 + \cdots \in S_{38}(\Gamma_0(288), \chi_{\text{triv}}).$$

Also let $H_{11}(z) = \sum_{n=1}^{\infty} c_{11}(n)q^n \in S_2(\Gamma_0(726))$ be the cusp form associated to the elliptic curve E_{11} (which has conductor 726). We wish to show that

$$\mathfrak{F}_{11}(z) \equiv \mathfrak{G}_{11}(z) \equiv \sum_{n=1}^{\infty} \chi_{12}(n) n^3 c_{11}(n) q^n \pmod{11}.$$

We have that

$$\Theta^3 H_{11} = \sum_{n=1}^{\infty} n^3 c_{11}(n) q^n$$

is congruent to a modular form in $S_{38}(\Gamma_0(726))$. In fact, more is true; it is congruent to a modular form in $S_{28}(\Gamma_0(6))$ (again, this can be verified using Sturm's theorem). Now, the χ_{12} quadratic twist

$$\left(\Theta^3 H_{11}\right)_{\chi_{12}} = \sum_{n=1}^{\infty} n^3 \left(\frac{12}{n}\right) c_{11}(n) q^n$$

is congruent to a modular form in $S_{28}(\Gamma_0(864))$.

Since $\mathfrak{F}_{11}(z) \in S_{28}(\Gamma_0(864))$ as well, we may check that

$$\sum_{n=1}^{\infty} n^3 \left(\frac{12}{n}\right) c_{11}(n) q^n \equiv \mathfrak{F}_{11}(z) \pmod{11}$$

using Sturm's theorem by checking that the first 4032 coefficients agree (mod 11). But as before, we need only check the coefficients of prime index, i.e., that

$$\ell^3 \left(\frac{12}{\ell}\right) c_{11}(\ell) \equiv \frac{1}{11} p\left(11 \cdot \frac{13(\ell^2-1)}{24} + \frac{13 \cdot 11 + 1}{24}\right) + \left(\frac{39}{\ell}\right) \ell^{13} \pmod{11}$$

for all primes $3 < \ell < 4032$.

Then to show that $\mathfrak{G}_{11} \equiv \sum_{n=1}^{\infty} \chi_{12}(n) n^3 c_{11}(n) q^n \pmod{11}$, we first note that they are both congruent to modular forms in $S_{38}(\Gamma_0(288))$. Thus as before, we need only check that

$$\ell^3 \left(\frac{12}{\ell}\right) c_{11}(\ell) \equiv \frac{1}{4 \cdot 11^2} p\left(11^2 \cdot \frac{23(\ell^2-1)}{24} + \frac{23 \cdot 11^2 + 1}{24}\right) + \left(\frac{-69}{\ell}\right) \ell^{18} \pmod{11}$$

for all primes $3 < \ell < 1824$. This completes the proof of the theorem. $\square$

4. Proof of Theorem 1.1 and Corollaries 1.2 and 1.3

Proof of Theorem 1.1. By Lemma 3.1 and Theorem 3.3, we have that

$$a_7(\ell^2 n) \equiv \ell \left(\frac{3}{\ell}\right) \left\{1 + \ell - \#E_7(\mathbb{F}_\ell) - \left(\frac{n}{\ell}\right)\right\} a_7(n) - \ell^3 a_7(n/\ell^2) \pmod{7}$$
$$b_7(\ell^2 n) \equiv \ell \left(\frac{3}{\ell}\right) \left\{1 + \ell - \#E_7(\mathbb{F}_\ell) - \ell^3 \left(\frac{-n}{\ell}\right)\right\} b_7(n) - \ell^3 b_7(n/\ell^2) \pmod{7}.$$

Then replacing n with $24n + 17$ and $24n + 23$, respectively and simplifying using Corollary 2.2 yields the results in (1).

Similarly, by Lemma 3.2 and Theorem 3.3, we have that

$$a_{11}(\ell^2 n) \equiv \ell^3 \left(\frac{3}{\ell}\right) \left\{1+\ell-\#E_{11}(\mathbb{F}_\ell) - \left(\frac{n}{\ell}\right)\right\} a_{11}(n) - \ell^7 a_{11}(n/\ell^2) \pmod{11}$$
$$b_7(\ell^2 n) \equiv \ell^3 \left(\frac{3}{\ell}\right) \left\{1+\ell-\#E_{11}(\mathbb{F}_\ell) - \ell^5\left(\frac{-n}{\ell}\right)\right\} b_{11}(n) - \ell^7 b_{11}(n/\ell^2) \pmod{11}$$

Then replacing n with $24n+13$ and $24n+23$, respectively, and simplifying using Corollary 2.3 yields the results in (2). □

Proof of Corollary 1.2. Replace n with $N\ell^2+\ell s+r$ in part (1) of Theorem 1.1 and note that

$$\frac{7^j(\ell^2 N+\ell s+r)}{\ell^2}+\beta_7(j,\ell^{-1})$$

cannot be an integer by condition (b). Therefore since $p(7^j n+\beta_7(j)) \equiv 0 \pmod{7^{\lfloor j/2\rfloor+1}}$ we have the result using the condition on ℓ, and (a). This proves (1), and the proof of (2) proceeds similarly.

To prove the statements regarding proportions, we must consider the associated Galois representation

$$\rho_{E_p,p}:\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to \mathrm{GL}_2(\mathbb{F}_p)$$

for $p\in\{7,11\}$. If we let N_p be the conductor of E_p, then it is well known that for any prime $\ell \nmid pN_p$, we have that

$$\operatorname{tr}\rho_{E_p,p}(\mathrm{Frob}_\ell)\equiv c_p(\ell)=1+\ell-\#E_p(\mathbb{F}_\ell) \pmod{p}$$
$$\det\rho_{E_p,p}(\mathrm{Frob}_\ell)\equiv \ell \pmod{p}.$$

Now recall a result of Serre (Proposition 19 of [8]), which says that for a prime $p\geq 5$ and subgroup $G\subseteq \mathrm{GL}_2(\mathbb{F}_p)$, if

(a) G contains an element s such that $\mathrm{tr}(s)^2-4\det(s)$ is a nonzero square in $\mathbb{F}_p$, and $\mathrm{tr}(s)\neq 0$,
(b) G contains an element s' such that $\mathrm{tr}(s')^2-4\det(s')$ is not a square in $\mathbb{F}_p$, and $\mathrm{tr}(s)\neq 0$,
(c) G contains an element s'' such that $u:=\frac{\mathrm{tr}(s'')^2}{\det(s'')}$ is distinct from $0,1,2,4$ (in $\mathbb{F}_p$), and $u^2-3u+1\neq 0$, and
(d) the map $\det: G\to\mathbb{F}_p$ is surjective,

then $G=\mathrm{GL}_2(\mathbb{F}_p)$. Thus by considering a few values of ℓ, one can easily check that $\rho_{E_p,p}:\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to\mathrm{GL}_2(\mathbb{F}_p)$ is surjective for $p=7,11$.

Now, to find the proportion of primes ℓ for which $\#E_p(\mathbb{F}_\ell)\equiv 1+\ell$, we may apply the Chebotarev Density Theorem, which implies that the desired proportion is simply the proportion of elements of $\mathrm{GL}_2(\mathbb{F}_p)$ with trace 0. The result follows easily, since $\#\mathrm{GL}_2(\mathbb{F}_7)=2016$, where 294 of these elements have trace 0 (and $\#\mathrm{GL}_2(\mathbb{F}_{11})=13200$, where 1210 elements have trace 0). □

Proof of Corollary 1.3. Similarly, replace n by $\ell N+r$ in part (1) of Theorem 1.1 and note that for j odd we have

$$\frac{7^j(\ell^2 N+\ell s+r)}{\ell^2}+\beta_7(j,\ell^{-1})$$

cannot be an integer since $24r + 13 \not\equiv 0 \pmod{\ell}$. Therefore since $p(7^j n + \beta_7(j)) \equiv 0 \pmod{7^{\lfloor j/2 \rfloor + 1}}$ we have the result using the conditions on ℓ and r. This proves (1) for odd j, and the proofs of (1) for even j and (2) proceed similarly.

The statements regarding proportions are also proved as in Corollary 1.2, noting that there are 287 elements of $\mathrm{GL}_2(\mathbb{F}_7)$ with trace ± 1, and 1199 elements of $\mathrm{GL}_2(\mathbb{F}_{11})$ with trace ± 1. □

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