Chapter 8:

#5. Let \((A_n : n = 1, 2, \cdots)\) be a sequence of measurable sets in a measure space \((\Omega, \mathcal{F}, \mu)\). Show that

\[
\liminf_{n \to \infty} \mu(A_n) \geq \mu(\liminf_{n \to \infty} A_n)
\]

Also show that the following inequality does not hold in the general measure space setting:

\[
\limsup_{n \to \infty} \mu(A_n) \leq \mu(\limsup_{n \to \infty} A_n)
\]

**Proof.** By Fatou lemma

\[
\liminf_{n \to \infty} \mu(A_n) = \liminf_{n \to \infty} \int I_{A_n} \, d\mu \\
\geq \int \liminf_{n \to \infty} I_{A_n} \, d\mu = \int I_{\liminf_{n \to \infty} A_n} \, d\mu = \mu(\liminf_{n \to \infty} A_n)
\]

We now consider the measure space \((\mathbb{R}, \mathcal{B}, \lambda)\), where \(\lambda\) is the one-dimensional Lebesgue measure. Let \(A_n = (n, \infty)\) \((n \geq 1)\). Then \(\lambda(A_n) = \infty\) for all \(n \geq 1\). On the other hand, \(\limsup_{n \to \infty} A_n = \phi\). Hence \(\lambda(\limsup_{n \to \infty} A_n) = 0\). So the second inequality does not hold in this case.

# 6. For the coin-flip probability space, Example 4 of Chapter 1, let \(X_n\) denote the indicator function of the event that the \(n^{th}\) flip is heads. Calculate \(\liminf_{n \to \infty} E(X_n)\) and \(E(\liminf_{n \to \infty} X_n)\).

**Solution.** This problem is to show that the inequality in Fatou lemma can not be replaced by equality in the general setting. Since \(X_n\) is indicator function, so is \(\liminf_{n \to \infty} X_n\). Further one can see that

\[
\liminf_{n \to \infty} X_n = I_{\{\text{only finite tails appear}\}}
\]

Switching heads and tails in Probelm 12 of Chapter 6 we have

\[
\liminf_{n \to \infty} X_n = 0 \quad a.s.
\]

So \(E(\liminf_{n \to \infty} X_n) = 0\).

On the other hand,

\[
E(X_n) = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}
\]

So \(\liminf_{n \to \infty} E(X_n) = \frac{1}{2}\)
#9. [Dominated Continuity of Measure Theorem]. Let \((A_n : n = 1, 2, \ldots)\) be a sequence of measurable sets in a measure space \((\Omega, \mathcal{F}, \mu)\). Show that if \(A = \lim_{n \to \infty} A_n\) exists and if
\[
\mu \left( \bigcup_{n=1}^{\infty} A_n \right) < +\infty \quad (*)
\]
then
\[
\mu(A) = \lim_{n \to \infty} \mu(A_n)
\]

**Proof.** In view of Problem 5, we see that the assumption (*) is crucial. Since 
\[
\lim_{n \to \infty} I_{A_n} = I_A
\]
\[
0 \leq I_{A_n} \leq I_{\bigcup_{k=1}^{\infty} A_k}
\]
and
\[
\int I_{\bigcup_{k=1}^{\infty} A_k} \, d\mu = \mu \left( \bigcup_{k=1}^{\infty} A_k \right) < +\infty
\]

By Dominated convergence theorem,
\[
\lim_{n \to \infty} \mu(A_n) = \lim_{n \to \infty} \int I_{A_n} \, d\mu = \int \lim_{n \to \infty} I_{A_n} \, d\mu = \int I_A \, d\mu = \mu(A)
\]

#13. Let \((X_1, X_2, \cdots)\) be a sequence of random variables, each with finite mean. Assume that \(\lim_{n \to \infty} X_n = 0\) almost surely and that \(\sup_n Var(X_n) < \infty\). Show that
\[
\lim_{n \to \infty} E|X_n| = 0
\]

**Proof.** By Theorem 12 we need only to show that \((X_1, X_2, \cdots)\) is uniformly integrable, i.e.,
\[
\lim_{c \to \infty} \sup_{n \geq 1} E \left( |X_n| I_{\{|X_n| \geq c\}} \right) = 0 \quad (1)
\]
First, by Chebyshev inequality
\[
P \{ |X_n - E(X_n)| \geq c \} \leq c^{-2} Var(X_n) \leq c^{-2} M
\]
where \(M = \sup_n Var(X_n) < \infty\). Therefore, we can choose \(c_0 > 0\) large enough, so that
\[
\sup_n P \{ |X_n - E(X_n)| \geq c_0 \} \leq \frac{1}{4}
\]
On the other hand, the assumption that \( \lim_{n \to \infty} X_n = 0 \) almost surely clearly implies that \( \sup_k |X_k| < \infty \) almost surely. Hence (why?) one can let \( c_o > 0 \) large enough so that

\[
P\{|X_n| \geq c_o\} \leq P\{\sup_k |X_k| \geq c_o\} \leq \frac{1}{4} \quad \forall n \geq 1
\]

Notice that by triangle inequality

\[
\{|E(X_n)| \geq 2c_o\} \leq \{|X_n - E(X_n)| \geq c_o\} \cup \{|X_n| \geq c_o\}
\]

By subadditivity,

\[
P\{|E(X_n)| \geq 2c_o\} \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2} < 1
\]

Since \( E(X_n) \) is not random, we must have

\[
|E(X_n)| \leq 2c_o \quad \forall n \geq 1
\]

Consequently,

\[
EX_n^2 = Var(X_n) + (E(X_n))^2 \leq M + 4c_o^2 \quad \forall n \geq 1
\]

So we have

\[
N \equiv \sup_n EX_n^2 < \infty
\]

By Cauchy-Schwarzs Inequality, for each \( c > 0 \),

\[
E\left(|X_n|I_{|X_n| \geq c}\right) \leq \left(EX_n^2\right)^{1/2}\left(EI_{|X_n| \geq c}\right)^{1/2} \\
= \left(EX_n^2\right)^{1/2}\left(P\{|X_n| \geq c\}\right)^{1/2} \leq \sqrt{N}\left(P\{|X_n| \geq c\}\right)^{1/2}
\]

Hence

\[
\sup_n E\left(|X_n|I_{|X_n| \geq c}\right) \leq \sqrt{N}\left(\sup_n P\{|X_n| \geq c\}\right)^{1/2}
\]

So (1) follows from the fact that

\[
\sup_n P\{|X_n| \geq c\} \leq P\{\sup_k |X_k| \geq c\} \longrightarrow 0 \quad (c \to \infty).
\]