Homework # 4

9.4*. Let \( m \geq 1 \) be a fixed but arbitrary integer.

\[
|EX1_{A_n}| \leq E|X|1_{\{|X| \leq m\}}1_{A_n} + E|X|1_{\{|X| > m\}}1_{A_n}
\]

\[
\leq mP(A_n) + E|X|1_{\{|X| > m\}}1_{A_n}
\]

Thus,

\[
\limsup_{n \to \infty} |EX1_{A_n}| \leq E|X|1_{\{|X| > m\}}
\]

It remains to show that

\[
\lim_{m \to \infty} E|X|1_{\{|X| > m\}} = 0
\]

First, notice that

\[
\lim_{m \to \infty} |X|1_{\{|X| > m\}} = 0 \quad a.s.
\]

By the bound \(|X|1_{\{|X| > m\}} \leq |X|\) and by dominated convergence theorem,

\[
\lim_{m \to \infty} E|X|1_{\{|X| > m\}} = 0.
\]

9.5* Clearly, \( Q(\Omega) = 1 \). According to Definition 2.3, p.8, all we need is to verify that for every countable, pair wise sequence \( \{A_n\} \) in \( A \)

\[
Q\left( \bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} Q(A_n)
\]

Indeed, for each \( m \geq 1 \), write \( B_m = \bigcup_{n=1}^{m} A_n \) and \( B_\infty = \bigcup_{n=1}^{\infty} A_n \). Notice that the disjointness leads to

\[
1_{B_m} = \sum_{n=1}^{m} 1_{A_n}
\]

Consequently,

\[
EX1_{B_m} = \sum_{n=1}^{m} EX1_{A_n} = \sum_{n=1}^{m} Q(A_n)
\]

It remains to show that

\[
\lim_{m \to \infty} EX1_{B_m} = EX1_{B_\infty}
\]

Indeed, it follows from the fact \( X1_{B_m} \uparrow X1_{B_\infty} \) and monotonic convergence (or dominated convergence).

9.8. (a). We follow the definition that

\[
E_QX^{-1} = \sup \left\{ E_QY; \ Y \text{ is simple and } Y \leq X^{-1} \right\}
\]
Let 
\[ Y = \sum_{k=1}^{n} a_k 1_{A_k} \]
where \( a_k \) are constant and \( A_k \in \mathcal{A} \) are pairwise disjoint. We assume \( Y \leq X \). Consequently, \( a_k \leq X(\omega)^{-1} \), or \( a_k X \leq 1 \) for \( \omega \in A_k \). Notice that

\[ E_Q Y = \sum_{k=1}^{n} a_k Q(A_k) \leq \sum_{k=1}^{n} E 1_{A_k} = \sum_{k=1}^{n} P(A_k) = P\left( \bigcup_{k=1}^{n} A_k \right) \leq 1 \]

Therefore \( E_Q X^{-1} \leq 1 < \infty \).

(b) For each \( A \in \mathcal{A} \), taking \( Y = X^{-1} 1_{A} \) in 9.7* we have \( R(A) = E_Q X^{-1} 1_{A} = EX^{-1} 1_{A} X = P(A) \).

9.11. By definition \( Var(X) = E[X - \mu]^2 \). By Expectation rule (Corollary 9.1) with \( h(x) = (x - \mu)^2 \) the right hand side is equal to

\[ \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx \]

9.16.
\[ EX^k = \int_{-\infty}^{\infty} x^k f(x) dx = \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} x^{k+\alpha-1} e^{-x} dx = \frac{\Gamma(k + \alpha)}{\Gamma(\alpha)} \]

By the relation \( \Gamma(\theta + 1) = \theta \Gamma(\theta) \) (\( \theta > 0 \)),
\[ EX = \frac{\Gamma(1 + \alpha)}{\Gamma(\alpha)} = \alpha, \quad EX^2 = \frac{\Gamma(2 + \alpha)}{\Gamma(\alpha)} = (\alpha + 1)\alpha \]

Thus,
\[ Var(X) = EX^2 - (EX)^2 = (\alpha + 1)\alpha - \alpha^2 = \alpha \]

9.18.
\[ P\{\mu - d\sigma < X < \mu + d\sigma\} = P\{|X - \mu| < d\sigma\} = 1 - P\{|X - \mu| \geq d\sigma\} \]

By Chebyshev inequality,
\[ P\{|X - \mu| \geq d\sigma\} \leq \frac{\sigma^2}{(d\sigma)^2} = \frac{1}{d^2} \]

Therefore,
\[ P\{\mu - d\sigma < X < \mu + d\sigma\} \geq 1 - \frac{1}{d^2} \]

9.19.
\[ P\{X > x\} = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-u^2/2} du \leq \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} \frac{u}{x} e^{-u^2/2} du \]
\[ = \frac{1}{x\sqrt{2\pi}} \int_{x}^{\infty} u e^{-u^2/2} du = \frac{1}{x\sqrt{2\pi}} e^{-x^2/2} \]