5.1 By monotonicity,

\[ P\{|X| \geq a\} \leq P\{g(|X|) \geq g(a)\} \leq \frac{E g(|X|)}{g(a)} \]

5.16. The correct form should be

\[ P\{X > i + j \mid X \geq i\} = P\{X > j\} \]

Indeed,

\[ P\{X > j\} = \sum_{k=j+1}^{\infty} (1-p)^k p = (1-p)^{j+1} p \sum_{k=0}^{\infty} (1-p)^k = (1-p)^{j+1} \]

Replacing \( j \) by \( i + j \),

\[ P\{X > i + j\} = (1-p)^{i+j+1} \]

Replacing \( j \) by \( i - 1 \),

\[ P\{X \geq i\} = P\{X > i - 1\} = (1-p)^i \]

Thus

\[ P\{X > i + j \mid X \geq i\} = \frac{P\{X > i + j, X \geq i\}}{P\{X \geq i\}} = \frac{P\{X > i + j\}}{P\{X \geq i\}} = (1-p)^{j+1} = P\{X > j\} \]

5.19. Intuitively, this follows from linearity of the expectation. On the other hand, justification is needed for the infinite summation. It is not hard if the monotonic convergence theorem (will be taught later) is allowed. Without monotonic convergence theorem, we need a delicate algebraic operation.

Notice that the random variable

\[ X = \sum_{n=1}^{\infty} 1_{A_n} \]

represents the number of happening events among \( A_1, \ldots, A_n, \ldots \). By definition,

\[ E\left( \sum_{n=1}^{\infty} 1_{A_n} \right) = E(X) = \sum_{k=0}^{\infty} k P\{X = k\} = \sum_{k=1}^{\infty} k P\{X = k\} \]

For any finite set \( I \subset \{1, 2, \ldots\} \), set

\[ C_I = \left( \bigcap_{n \in I} A_n \right) \cap \left( \bigcap_{n \notin I} A_n^c \right) \]
Then for any finite sets $I, J \subset \{1, 2, \cdots\}$ with $I \neq J$, $C_I \cap C_J = \phi$.

In addition, for any $n \geq 1$,

$$
\bigcup_{\text{finite } I: n \in I} C_I = A_n
$$

Further, for any $k \geq 1$, by countable additivity of the probability

$$
P\{X = k\} = P\left( \bigcup_{I: \#(I) = k} C_I \right) = \sum_{I: \#(I) = k} P(C_I)
$$

Hence,

$$
E\left( \sum_{n=1}^{\infty} 1_{A_n} \right) = \sum_{k=1}^{\infty} k \sum_{I: \#(I) = k} P(C_I)
$$

Notice that

$$
k \sum_{I: \#(I) = k} P(C_I) = \sum_{I: \#(I) = k} \sum_{n: n \in I} P(C_I)
$$

Hence,

$$
E\left( \sum_{n=1}^{\infty} 1_{A_n} \right) = \sum_{k=1}^{\infty} \sum_{I: \#(I) = k} \sum_{n: n \in I} P(C_I) = \sum_{n=1}^{\infty} \sum_{\text{finite } I: n \in I} P(C_I)
$$

$$
= \sum_{n=1}^{\infty} P\left( \bigcup_{\text{finite } I: n \in I} C_I \right) = \sum_{n=1}^{\infty} P(A_n)
$$

where the third equality follows from countable additivity of probability.

5.20. Notice that

$$
X = \sum_{n=0}^{\infty} \mathbf{1}_{\{X > n\}}
$$

The requested conclusion follows from the conclusion of 5.19.

**Alternative proof**

$$
E(X) = \sum_{k=1}^{\infty} k P\{X = k\} = \sum_{k=1}^{\infty} k \left( P\{X > k - 1\} - P\{X > k\} \right)
$$

By the fact

$$
\sum_{k=1}^{n} k \left( P\{X > k - 1\} - P\{X > k\} \right) = \sum_{k=0}^{n-1} (k+1) P\{X > k\} - \sum_{k=1}^{n} k P\{X > k\}
$$

$$
= -nP\{X > n\} + \sum_{k=0}^{n-1} P\{X > k\}
$$
Whn $n \to \infty$, the both sides converges or diverges simultaneously. When converging, the left hand gives that $E(X) < \infty$. By (the spirit of) Markov inequality,

$$nP\{X > n\} \leq \sum_{k=n+1} kP\{X = k\} \to 0 \quad (n \to \infty)$$

So we have

$$E(X) = \sum_{k=0}^{\infty} P\{X > k\}$$