GENERIC FLATNESS AND THE JACOBSON CONJECTURE

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Abstract. A result of Artin, Small, and Zhang is used to show that a noetherian algebra over a commutative, noetherian Jacobson ring will be Jacobson if the algebra possesses a locally finite, noetherian associated graded ring. This result is extended to show that if an algebra over a commutative noetherian ring has a locally finite, noetherian associated graded ring, then the intersection of the powers of the Jacobson radical is nilpotent. The proofs rely on a weak generalization of generic flatness and some observations about G-rings.

1. Introduction

The Jacobson radical was introduced in a 1945 paper of Jacobson in which he proved that for one-sided noetherian rings that there exists an ordinal \( \alpha \) such that \( J^\alpha = (0) \) [9, Theorem 11]. He also conjectured that \( \omega \) would suffice: that \( J(R)^\omega = \bigcap_{n=1}^\infty J(R)^n \) would equal the zero ideal.

Herstein [8] gave the first counterexample to this original conjecture: a class of two-by-two upper triangular matrices in which the upper-left entry is taken from a local integral domain, and the upper- and lower-right entries are taken from the domain’s field of fractions.

In 1967 Jategaonkar found a class of examples which are not right noetherian, in which all left ideals are principal, and that have unique maximal ideals; these rings have \( \bigcap_{n=1}^\infty J^n \neq 0 \) [10], [11]. Additionally, these rings are domains, so non-zero ideals are not nilpotent; therefore, the intersections of the powers of their Jacobson radicals will not be nilpotent.

In light of these examples in one-sided noetherian rings, Jacobson’s conjecture has been refined to the case where the ring is both left and right noetherian.

**Conjecture 1.1** (Jacobson Conjecture). Let \( R \) be a left and right noetherian ring, and let \( J \) be the Jacobson radical of \( R \), then \( \bigcap_{n=1}^\infty J^n = (0) \).

It is immediate from the Krull Intersection Theorem that this holds for all commutative rings. No general result exists for all noetherian rings, but it has verified for several special cases, including fully bounded noetherian (FBN) rings [2, 7.5] and rings of Krull dimension 1 [2, 5.13]. It has also been shown for rings satisfying a polynomial identity [19, 5.2.26].

In Herstein’s example, the intersection of the powers of the Jacobson radical is nilpotent. A problem closely related to the Jacobson Conjecture...
would be to determine when the intersection of the powers of the radical in any (one-sided) noetherian ring is nilpotent. As Jategoankar’s examples illustrate, there are noncommutative integral domains where the intersection of the powers of the radical is neither zero nor nilpotent.

Some rings have a nilpotent or nil Jacobson radical (in a noetherian ring, nil ideals are always nilpotent [17, 2.3.7]). This occurs when the Jacobson radical is contained in the prime radical, and this is the defining feature of a Jacobson ring.

**Definition 1.2.** A ring $R$ is a Jacobson ring if every prime ideal is an intersection of primitive ideals. Equivalently, for every prime ideal $P$ of $R$, the Jacobson radical, $J(R/P)$, of the factor ring is zero.

**Remark 1.3.** When the Jacobson radical is contained in the prime radical, then the Jacobson radical is also contained in each prime ideal of the ring. Whenever the Jacobson radical is contained in the prime radical of all homomorphic images, the ring is Jacobson.

**Example 1.4** (Amitsur). Let $k$ be an uncountable field. Any countably generated algebra over $k$ is a Jacobson ring [17, 9.1.8].

Jacobson rings were first defined and named in a 1951 paper of Krull in which he proved the initial results about these rings [14]. He showed that if $R$ is a commutative Jacobson ring, then every finite ring extension of $R$ is also a Jacobson ring. Similar work was also done by Goldman [5], in which he used the name Hilbert ring to emphasize the connection with Hilbert’s Nullstellensatz. Skew polynomial extensions of commutative Jacobson rings are Jacobson [6, Theorem 4.3], and if $K$ is any Jacobson ring, then $K[y]$ will also be a Jacobson ring [22]. Enveloping algebras of finite dimensional Lie algebras are Jacobson. Additionally, a ring which is a finite module over a Jacobson ring will be Jacobson [3, Theorem 1]. Most classes of affine noetherian rings are known to be Jacobson or else they can be localized to obtain a Jacobson ring. More specifically, there is no known example of a noetherian ring which is a finitely generated algebra over a field and which fails to be a Jacobson ring.

This paper uses a result of Artin, Small, and Zhang [1] to describe a class of algebras which are Jacobson rings and a larger class for which the Jacobson Conjecture holds.

The algebra must have an associated graded ring which satisfies certain conditions, so we make the following definitions:

**Definition 1.5.** A filtration of a ring $R$ is a family of subgroups of $R$, $F_0R \subset F_1R \subset \cdots$ such that $F_iR F_jR \subset F_{i+j}R$ and $\bigcup F_n = R$. The associated graded ring of $R$, denoted $\text{gr} R$, is the direct sum of the quotients $R_n = F_nR/F_{n-1}R$. The indices are non-negative integers.

**Definition 1.6.** A ring which has a filtration such that the associated graded ring is noetherian is called filtered noetherian.
Remark 1.7. All filtered noetherian rings are noetherian rings [17, 1.6.9].

Definition 1.8. Let $A$ be a filtered $R$-algebra with associated graded ring $\text{gr}A$. We say that $\text{gr}A$ is locally finite if $A_n$ is a finite $R$-module for each $n$.

Definition 1.9. Let $R$ be a noetherian ring, and let $A$ be a filtered noetherian $R$-algebra such that $F_0A = R$ and $\text{gr}A$ is locally finite. Then we shall say that $A$ is a well-filtered $R$-algebra.

This paper proves two results about well-filtered $R$-algebras:

Theorem 5.1. Let $R$ be a commutative Jacobson ring, and let $A$ be a well-filtered, affine $R$-algebra. Then $A$ is a Jacobson ring.

Theorem 6.3. Let $R$ be a commutative, noetherian ring. Let $A$ be a well-filtered, affine $R$-algebra. Then the intersection of the powers of the Jacobson radical of $A$ is nilpotent.

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2. Noncommutative Nullstellensatz

Krull forged a link between Jacobson rings and noncommutative generalizations of the Nullstellensatz.

Definition 2.1. Let $R$ be an algebra over a field $k$. If for each simple $R$-module, $M$, the endomorphism ring $\text{End}(M)$ is algebraic over $k$, then we say that $R$ satisfies the Nullstellensatz over $k$.

The relationship between this idea and the concept of Jacobson rings is so strong that in [17] they restrict the definition of algebras satisfying the Nullstellensatz to Jacobson algebras satisfying definition 2.1. The commutative formulation of the Nullstellensatz given in [4, 4.19], which phrases the result in terms of the dimension of a factor by a maximal ideal, only applies to algebras over Jacobson rings.

Combining the techniques mentioned previously with the observation that for a field $k$ every graded noetherian module is generically flat over $k[t]$, Artin, Small, and Zhang proved the following:

Theorem 2.2 (Artin, Small, Zhang). Let $A$ be a well-filtered algebra over a field $k$. Then $A$ is a Jacobson algebra which satisfies the Nullstellensatz [1, Theorem 0.4].

Many classes of rings are filtered noetherian, including universal enveloping algebras of finite dimensional Lie algebras, quantum groups [16], rings which are finite modules over a noetherian central subring [23, Lemma 6.13], and homomorphic images of graded noetherian rings. Of course, the homomorphic images of graded rings might no longer be graded.
Not all noetherian rings possess a filtration such that the associated graded ring is noetherian. In 1997 Stephenson and Zhang proved that there exist finitely generated noetherian algebras for which no finite dimensional filtration leads to a right noetherian associated graded ring. Indeed, they showed that any noetherian ring which exhibits exponential growth cannot be filtered noetherian [21, Theorem 0.1]. Additionally, Stafford and Zhang [20] have developed a method for constructing rings which are noetherian PI algebras and finitely generated (as modules) over a commutative (but not central) subring but which are not filtered noetherian.

Well-filteredness is conserved in certain common situations. All noetherian graded rings are filtered noetherian rings with respect to the filtration derived from the grading. The following two lemmas are easily proven:

**Lemma 2.3.** Let \( R \) be a commutative, noetherian ring, and let \( A \) be a well-filtered \( R \)-algebra. Let \( I \) be an ideal of \( A \). Then \( A/I \) is a well-filtered \( R/(R \cap I) \)-algebra.

**Lemma 2.4.** Let \( R \) be a commutative, noetherian ring, and let \( A \) be a well-filtered \( R \) algebra. Let \( S \) be a multiplicatively closed set of \( R \). Then \( A_S \) is a well-filtered \( R_S \)-algebra.

## 3. Generic Flatness and Rational Closure

The stated results (and others) about Jacobson rings and the Nullstellensatz have been proved using a property known as generic flatness. This property is more descriptively referred to as generic freeness. The definition of this property relies on the following:

**Definition 3.1.** Let \( R \) be a commutative integral domain, and let \( c \) be a non-zero element of \( R \). A simple localization of \( R \) at \( c \), denoted \( R_c \), is \( R[c^{-1}] \), the ring obtained by inverting \( c \).

We now can define generic freeness.

**Definition 3.2.** Let \( K \) be an integral domain, and let \( R \) be a \( K \)-algebra. Then \( R \) satisfies generic freeness over \( K \) if for each finitely generated \( R \)-module \( M \), there exists a non-zero \( c \) in \( K \) such that the localization \( M_c = M \otimes_R K_c \) is a free \( K_c \)-module.

The term generic flatness is used when \( M_c \) is a flat \( K_c \)-module. The term generic projectivity is defined similarly. The distinction between generic freeness and generic flatness is important. As noted in [1], there is no simple localization that will make \( k(x) \) into a free \( k[x] \)-module, but it is a flat \( k[x] \)-module.

The notion of generic freeness has been used in algebraic geometry and was applied to noncommutative Nullstellensatz-type situations by Quillen [18]. He used generic freeness to prove that for a universal enveloping algebra over a base field \( k \) (and for algebras with similar filtered-graded properties) that the endomorphism ring of a simple module is algebraic over \( k \). Rational
closure is a very weak generalization of generic freeness. Typically, we require that the inversion of a single element gives an algebra which is free, flat, or projective over the simple localization of the base ring. Here all that is required is that the algebra possess an associated graded ring which is torsion free over the localization of the base ring. When this is applied to filtered noetherian rings, we know that such a simple localization exists. This construction, especially in the case of prime rings over integral domains, assures that zero divisors will not cause difficulties. Without this assurance, some constructions might give us the zero ring or zero module.

**Definition 3.3.** We say that \( R \subset S \) is rationally closed if for any \( r \in R \) with an inverse in \( S \), then \( r^{-1} \) is already in \( R \).

A rationally closed subring is also called a full subring [15].

**Lemma 3.4** (Rational Closure Lemma). Let \( R \) be a commutative noetherian ring, and let \( A \) be a well-filtered, prime, \( R \)-algebra. Then there exists an \( 0 \neq r \in R \) such that \( R[r^{-1}] \) is rationally closed in \( A[r^{-1}] \) and \( A[r^{-1}] \) is a well-filtered \( R[r^{-1}] \)-algebra.

**Proof.** There are two cases to consider, depending on whether or not \( \text{gr} A \) has \( R \)-torsion. Because \( A \) is prime, its center is an integral domain, and \( R \) is an integral domain, so it is proper to discuss \( R \)-torsion (rather than torsion with respect to the regular elements of \( R \)).

In the first case, assume that \( \text{gr} A \) is \( R \)-torsionfree. In this case \( R \) is rationally closed in \( A \). The proof follows by contradiction: assume that there exists an \( s \in R \) such that \( s^{-1} \) is an element of \( A \) but that \( s^{-1} \) is not an element of \( R \). Then there must be an index \( i > 0 \) such that \( s^{-1} \in F_i A \setminus F_{i-1} A \) (and by assumption \( s^{-1} \) is not a member of \( F_0 A = R \)).

Consider \( \bar{s} \) and \( s^{-1} \), the images of \( s \) and \( s^{-1} \) in \( \text{gr} A \). By construction, \( s \in R = F_0 A \), so \( \bar{s} \in A_0 \). And by assumption \( s^{-1} \in A_i \) for some \( i > 0 \). The product \( \bar{s}s^{-1} \) is non-zero because \( \text{gr} A \) has no \( R \)-torsion. Because of the grading \( \bar{s}s^{-1} \in A_i \) for some \( i > 0 \). However, the choice of \( s \) and \( s^{-1} \) implies that \( \bar{s}s^{-1} = \bar{1} \), which is an element of \( A_0 \). Therefore the assumption that \( s^{-1} \in F_i A \setminus F_{i-1} A \) for some \( i > 0 \) can not be true, and \( s^{-1} \) must already be an element of \( R \).

In the second case, we consider what happens when \( \text{gr} A \) has \( R \)-torsion. By assumption, \( A \) has a filtration \( \{ F_n A \} \) such that it has an associated graded ring \( \text{gr} A = \oplus F_i A / F_{i-1} A \) which is noetherian. Since the \( R \)-torsion submodule \( T(\text{gr} A) \) is an ideal of \( \text{gr} A \), it is finitely generated. For each of the generators of this ideal, select an \( r_i \) that annihilates it. The product \( r = \prod r_i \) will be non-zero (\( R \) is an integral domain), and it will annihilate \( T(\text{gr} A) \). Localize at \( r \), and consider \( A[r^{-1}] \) as a \( R[r^{-1}] \)-algebra.

Because the powers of \( r \) is a multiplicatively closed set and a right denominator set of \( A \), Lemma 2.4 shows that \( A[r^{-1}] \) is a well-filtered \( R[r^{-1}] \)-algebra.

We must also confirm that \( \text{gr} A[r^{-1}] \) is \( R[r^{-1}] \)-torsion free. If \( \text{gr} A[r^{-1}] \) had \( R[r^{-1}] \)-torsion, then it would also have \( R \)-torsion (An element in \( R[r^{-1}] \)
which kills something in \( \text{gr}A[r^{-1}] \) can be multiplied by a suitably large power of \( r \) to give an element of \( R \) which would kill that element of \( \text{gr}A[r^{-1}] \). If \( a \in \text{gr}A[r^{-1}] \) has \( R \)-torsion, then by our choice of \( r \), \( ra = 0 \). However, \( a \) must in fact be zero, as \( a = 1 \cdot a = r^{-1}ra = r^{-1} \cdot 0 = 0 \). Thus, \( \text{gr}A[r^{-1}] \) does not have \( R \)-torsion, so it is also \( R[r^{-1}] \)-torsion free.

The same argument that completed the first case applies here, showing that \( R[r^{-1}] \) is rationally closed in \( A[r^{-1}] \). □

**Lemma 3.5** (Change of Base Ring Lemma). *Let \( R \) be a commutative, noetherian integral domain, and let \( A \) be a well-filtered \( R \)-algebra. If \( Q(R) \), the quotient field of \( R \), is contained in \( A \), then \( A \) is a well-filtered \( Q(R) \)-algebra.*

**Proof.** Let \( S = R \setminus \{0\} \). By Lemma 2.4, \( A_S \) will be a well-filtered \( R_S = Q(R) \)-algebra. Since, by assumption, \( Q(R) \subset A, A_S = A \), and \( A \) is a well-filtered \( Q(R) \)-algebra. □

4. **G-rings**

Several of the following results are proved for G-rings because in this class of rings there are only finitely many prime ideals of rank one (Lemma 4.3). While all noetherian rings contain a finite number of minimal primes [17, 2.2.16b], this statement about G-rings is somewhat stronger. Because G-rings are prime rings, \((0)\) is the unique, minimal prime in a G-ring.

Recall that a prime ring \( A \) is said to be a *G-ring* when the intersection of the non-zero primes of \( A \) is non-zero. A prime ideal \( P \) of \( A \) is said to be a *G-ideal* when \( A/P \) is a G-ring. This is equivalent to saying that a prime ideal \( P \) of \( A \) is a G-ideal when the primes properly containing \( P \) intersect in an ideal which properly contains \( P \). Maximal ideals are always G-ideals.

A commutative integral domain \( K \) is a G-ring if there exists a \( 0 \neq c \in K \) such that \( K_c \) is a field [17, 9.3.9], [13, section 1–3]. A prime PI ring \( R \) is a G-ring if there exists a \( 0 \neq c \in Z(R) \) such that \( R_c = Q(R) \) [17, 13.9.12]. Furthermore, any G-ring which is also a Jacobson ring will have its Jacobson radical equal zero. Additionally the following lemma shows a close relationship between the G-ideals and the collection of all prime ideals.

**Lemma 4.1.** In a noetherian ring every prime ideal is the intersection of G-ideals.

This implies that the intersection of the G-ideals of a ring is contained in the intersection of all its prime ideals.

The property of being a G-ring is preserved under localization.

**Lemma 4.2.** Let \( R \) be a G-ring and let \( S \) be a right denominator set of \( R \). Then \( R_S \) is a G-ring.

**Proof.** Since \((0)\) is a prime ideal of \( R \), \((0)\) is a prime ideal of \( R_S \) [7, 9.22]. Thus it is necessary to show that in \( R_S \) the intersection of the non-zero primes is non-zero. Let \( \{P_i\}_{i \in I} \) be the collection of non-zero primes of \( R_S \) for some index set \( I \). By [7, p155] for each \( P_i \subset R_S \) we can construct the
contraction $P'_i \subset R$ defined by $P'_i = \{ r \in R | r \cdot 1^{-1} \in P_i \}$. By [7, 9.20] each of these sets is a prime ideal of $R$.

Because G-rings are prime, the right denominator set is regular [7, 9.21], so the map $r \mapsto r \cdot 1^{-1}$ from $R$ to $R_S$ defined by the localization is an injection. Thus $P_i \cap R = P'_i$.

The set $(\cap_{i \in I} P_i) \cap R$ is contained in the intersection of the non-zero primes of $R_S$. This set is equal to the intersection of non-zero primes of $R$, so it is non-zero (because $R$ is a G-ring). Therefore, the intersection of the non-zero primes of $R_S$ is non-zero, and $R_S$ is a G-ring. □

**Lemma 4.3.** Let $R$ be a noetherian G-ring. Then $R$ has a finite number of primes of rank one.

**Proof.** In a noetherian ring, any intersection of primes can be written as an intersection of a finite number of primes.

In a G-ring, the intersection of all the primes containing $(0)$ is a non-zero ideal; call it $I$. It can be written as an intersection of a finite number of non-zero primes and can be refined so that each of the primes is of rank one: if a prime of higher rank appears on the list, then replace it with the rank one primes that it contains. (This will not change $I$ because it equals the intersection of all the non-zero primes, including those of rank one.) Further refine this list so that no prime appears on the list more than once on this finite list of non-zero primes. Call these primes $P_1, P_2, \ldots, P_k$.

These are, in fact, all of the rank one primes. Assume that there is another rank one prime $Q$ which is not on this list. Consider the product $\prod P_1 \cdot P_2 \cdots P_k$. Because $Q$ is a prime ideal, if this product is contained in $Q$, then $Q$ must be one of the $k$ factors that make up the product. Because $\prod_{n=1}^{k} P_n$ is contained in each of the $P_i$, $\prod_{n=1}^{k} P_n$ is contained in $I$. By construction $I$ is contained in every prime in $R$, so $I$ is contained in $Q$. This implies that $\prod_{n=1}^{k} P_n \subset I \subset Q$, contrary to the assumption that $Q$ is not one of the rank one primes. □

5. **Jacobson Rings**

**Theorem 5.1.** Let $R$ be a commutative Jacobson ring, and let $A$ be a well-filtered, affine $R$-algebra. Then $A$ is a Jacobson ring.

Arguably, this is the most general situation in which it has been shown that a ring extension of a Jacobson ring is also Jacobson. Its proof relies on the Prime Avoidance Lemma and a version of the Principal Ideal Theorem.

The Principal Ideal Theorem is a well-known result from commutative algebra; in its most general form states that a prime ideal in a commutative ring which is minimal over $x_1, x_2, \ldots, x_c$ will have rank at most $c$ [4, Theorem 10.2]. Jategaonkar gives a partial extension of this result to non-commutative rings; he proved that in a noetherian ring $R$ that for any non-zero $x \in R$, if $xR = Rx$, then any prime ideal of $R$ which is minimal over $x$ has rank at most one [12]. Although this observation is limited by its requirement that
the primes be minimal over elements which are normalizing, it has value in situations which involve lifting results about the ideals in a central subring. This is often combined with the Prime Avoidance Lemma [4, Lemma 3.3], [13, p55], which states that an ideal contained in a finite union of prime ideals in a commutative ring must be completely contained in (at least) one of those ideals.

Proof. The method of proof relies on considering reductions of $A$ by an arbitrary prime ideal and by a $G$-ideal.

The first reduction is by a prime ideal. Were $A$ not a Jacobson ring then there would exist a prime ideal $Q$ of $A$ such that the Jacobson radical of $A/Q$ was non-zero. Demonstrating that $J(A/Q)$ is zero for all primes $Q$ of $A$ shows that $A$ is a Jacobson ring.

Let $Q$ be a prime ideal of $A$, and consider $A/Q$ as a $R/R \cap Q$-algebra. Because $A/Q$ is prime, $R/R \cap Q$ is an integral domain, and $R \cap Q$ is a prime ideal of $R$. Because $R$ is (by assumption) a Jacobson ring, $J(R/R \cap Q) = 0$. By Lemma 2.3, $A/Q$ will be a well-filtered $R/R \cap Q$-algebra. Thus, after a prime reduction, $A/Q$ is a prime ring which is a well-filtered $R/R \cap Q$-algebra, and $R/R \cap Q$ is a prime, commutative, noetherian, Jacobson ring.

Thus, proving the theorem is reduced to showing that the Jacobson radical is always zero in all prime rings which are well-filtered algebras over a prime, commutative, noetherian, Jacobson ring. Therefore, let $R$ be a prime, commutative, noetherian, Jacobson ring, and let $A$ be a prime ring which is a well-filtered $R$-algebra. The goal is to show that the Jacobson radical of $A$ is zero.

This is accomplished though considering a G-reduction of the prime ring. Let $P$ be a G-ideal of $A$. If for every such $P$ the Jacobson radical of $A/P$ is zero, then $J(A)$ is contained in the intersection of the G-ideals of $A$. By the first reduction, $A$ is a prime ring, so $(0)$ is a prime ideal of $A$. By Lemma 4.1, $(0)$ is an intersection of G-ideals. Therefore, demonstrating that for each G-ideal $P$ of $A$ that the Jacobson radical of $A/P$ is zero is sufficient to prove the theorem.

The analysis that shows that the hypotheses are still satisfied after the reduction by a prime ideal also holds after reducing by a G-ideal because G-ideals are prime (by definition). For the rest of the proof let $R$ be a prime, commutative, noetherian, Jacobson ring, and let $A$ be a G-ring which is a well-filtered $R$-algebra.

The easiest case to consider is when $Q(R)$, the quotient field of $R$ is contained in $A$. By applying the Change of Base Ring Lemma (Lemma 3.5), $A$ is a well-filtered algebra over a field. Theorem 2.2 applies to this case, and $A$ is a Jacobson ring. By assumption $A$ is a prime ring, so $J(A) = (0)$.

The rest of the proof shows that there is no other case, that $R$ must be a field. This proceeds by contradiction. Assume that $R$ is not a field.

In this case, the rational closure Lemma can be applied, ensuring that there exists a $0 \neq c \in R$ such that $A[c^{-1}]$ is a well-filtered $R[c^{-1}]$-algebra.
Lemma 4.2 shows that $A[c^{-1}]$ is still a G-ring. Because $R[c^{-1}]$ is the homomorphic image of a polynomial ring in one variable over $R$, $R[c^{-1}]$ will be a Jacobson ring [22, p307].

By Jategaonkar’s version of the Principal Ideal Theorem, any non-unit in $R[c^{-1}]$ is contained in a prime of $A[c^{-1}]$ of rank one. Because $A[c^{-1}]$ is a G-ring, there are a finite number of such primes (Lemma 4.3). The intersection of any of these rank one primes of $A[c^{-1}]$ with $R[c^{-1}]$ will yield an ideal in $R[c^{-1}]$. Since all proper ideals consist of non-units, each ideal of $R[c^{-1}]$ is a Jacobson ring [22, p307].

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The intersection of any of these rank one primes of $A[c^{-1}]$ with $R[c^{-1}]$ will yield an ideal in $R[c^{-1}]$. Since all proper ideals consist of non-units, each ideal of $R[c^{-1}]$ is a Jacobson ring [22, p307]. Specifically, all of the maximal ideals of $R[c^{-1}]$ can be represented this way.

Invoking the Prime Avoidance Lemma shows that any maximal ideal of $R[c^{-1}]$ is contained in a finite union of ideals which are intersections of rank one primes of $A[c^{-1}]$ with $R[c^{-1}]$. Because there are only a finite number of rank one primes of $A[c^{-1}]$, there can only be a finite number of maximal ideals of $R[c^{-1}]$. Therefore, $R[c^{-1}]$ is a G-ring.

However, a commutative, Jacobson G-ring is a field, so $R[c^{-1}]$ is a field. Because inverting one element of $R$ gives a field, this implies that $R$ is a G-ring. Again, we have a commutative, Jacobson G-ring. From this we conclude that $R$ is a field—contrary to the assumption. □

6. The Jacobson Conjecture

The following results extend the classes of prime rings for which the Jacobson conjecture is known to hold. Additionally, they can also be applied in a weaker form to rings which are not prime, showing in those cases that the intersection of the powers of the Jacobson radical is a nilpotent ideal.

To use rational closure to prove Theorem 6.3, we require the following lemma:

**Lemma 6.1.** Let $A$ be a prime noetherian ring, and let $M$ be a centrally generated proper ideal of $A$. Then $\cap_{n=1}^{\infty} M^n = 0$.

**Proof.** Let $\Delta = \cap_{n=1}^{\infty} M^n$. Let $S = \{1 - m | m \in MA\}$. Since $M$ is centrally generated, the Artin-Rees property [17, 4.2.2] applies, and there exists an integer $N$ such that $\Delta \cap M^N \subset \Delta M$. By the construction of $\Delta$, $\Delta = \Delta \cap M^N$, and because $\Delta$ is an ideal, $\Delta M \subset \Delta$. This implies that $\Delta \subset \Delta M \subset \Delta$, and $\Delta M = \Delta$. Localizing at $S$ gives $\Delta_S M_S = \Delta_S$ ($S$ is a right denominator set by [17, 4.2.9]). Again, because the Artin-Rees property applies to $M$ [17, 4.2.9], $M_S \subset J(A_S)$. By Nakayama’s Lemma, this implies that $\Delta_S = 0$. Because $A$ is a prime ring, the right denominator set $S$ consists entirely of regular elements [7, 9.21], so $ass(A_S) = 0$ and $\Delta = 0$. □

**Lemma 6.2.** Let $R$ be a commutative, noetherian ring. Let $A$ be a well-filtered affine $R$-algebra which is a G-ring. Then the intersection of the powers of the Jacobson radical of $A$ is zero.
Proof. If \( R \) is a field, then \( A \) is a well-filtered algebra affine over a field, so by Theorem 2.2, \( A \) is a prime Jacobson ring, and \( J(A) \) is zero. Similarly, if \( Q(R) \) is contained in \( A \), then by the Change of Base Ring Lemma (Lemma 3.5), \( A \) will also be a filtered noetherian ring over a field and Theorem 2.2 applies again.

When \( R \) is not a field, Jategaonkar’s noncommutative version of the Principal Ideal Theorem provides for any non-unit \( c \in R \) a rank one prime in \( A \) which contains \( c \). Since \( A \) is a G-ring, there are only finitely many such primes (Lemma 4.3). Denote this collection of primes by \( \{ P_j \} \). Intersection with \( R \) provides a collection of ideals \( \{ P_j \cap R \} \) in \( R \). Some of these are, in fact, maximal ideals of \( R \). Let \( I \triangleleft R \); since \( I \) consists of non-units of \( R \), each element of \( I \) is contained in one (or more) of the \( P_j \cap R \). By the Prime Avoidance Lemma, \( I \) must be entirely contained in one of the \( P_j \cap R \), and \( \{ P_j \cap R \} \) are maximal ideals of \( R \). Thus, there exists a maximal ideal \( M \triangleleft R \) such that \( MA \neq A \) and \( MA \) is, of course, centrally generated (by elements in \( R \)).

\( A/MA \) is an affine algebra over the field \( R/M \). By Lemma 2.3 it is a well-filtered algebra over \( R/M \), and by Theorem 2.2, it is a Jacobson ring. By definition this implies that in each of its prime factors the Jacobson radical is zero, so \( J(A/MA) \) is contained in the prime radical of \( A/MA \). Because the prime radical is a nilpotent ideal in a noetherian ring \( \text{[17, 0.2.6 and 2.3.7]} \), the Jacobson radical of \( A/MA \) is nilpotent. Since there exists an integer \( t > 0 \) such that \( J(A/MA)^t = 0 \), this implies that \( J(A)^t \subset MA \).

Thus, \( \cap_{i=1}^\infty J(A)^i \subset \cap_{j=1}^\infty M^jA \). Since \( MA \) is generated by elements of \( M \), which is in the center of \( A \), \( MA \) is a centrally generated ideal. Because \( A \) is a prime noetherian ring (Remark 1.7) and \( MA \) is a proper ideal of \( A \), Lemma 6.1 applies, and the intersection of the powers of the Jacobson radical of \( A \) is zero. \( \square \)

**Theorem 6.3.** Let \( R \) be a commutative, noetherian ring. Let \( A \) be a well-filtered, affine \( R \)-algebra. Then the intersection of the powers of the Jacobson radical of \( A \) is nilpotent.

**Proof.** Let \( P \) be a G-ideal of \( A \). Then by Lemma 2.3 \( A/P \) will be an algebra over \( R/(R \cap P) \) which satisfies the hypotheses of Lemma 6.2. Any element of \( A \) which lies in the intersection of the powers of \( J(A) \) must be an element of \( P \). Otherwise this element’s image in \( A/P \) would lie in the intersection of the powers of \( J(A/P) \) and would be non-zero, contrary to Lemma 6.2. This reasoning holds for any G-ideal of \( A \), so \( \cap_{n=1}^\infty J(A)^n \) is contained in the intersection of the G-ideals of \( A \). By Lemma 4.1, the intersection of the G-ideals will be contained in the intersection of all the prime ideals. Since \( A \) is a noetherian ring (Remark 1.7), the prime radical of \( A \) is a nilpotent ideal \( \text{[17, 0.2.6, 2.3.7]} \), the intersection of the powers of the Jacobson radical of \( A \) must be nilpotent. \( \square \)
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