1. The trapezoidal method for solving the IVP \( y'(x) = f(x, y(x)), \ x \in [a, b], y(x_0) = y_0, \ x_0 \in [a, b], \) is defined as

\[
\eta_i+1 = \eta_i + \frac{h}{2} [f(x_i, \eta_i) + f(x_{i+1}, \eta_{i+1})], \quad i = 0, 1, 2, \ldots,
\]

with \( \eta_0 = y_0, \ x_i = x_0 + ih, \ h > 0. \) Prove that the method is globally convergent, and the global rate of convergence is second order, assuming that \( y \in C^3[a, b]. \)

Hint: We did this in class. Just fill in the missing details concerning truncation error and the induction proof establishing the estimate of the error.

2. The implicit midpoint method for solving the IVP \( y'(x) = f(x, y(x)), \ x \in [a, b], y(x_0) = y_0, \ x_0 \in [a, b], \) is defined as

\[
\eta_i+1 = \eta_i + hf\left(x_i + \frac{h}{2}, \frac{\eta_i + \eta_{i+1}}{2}\right), \quad i = 0, 1, 2, \ldots,
\]

with \( \eta_0 = y_0, \ x_i = x_0 + ih, \ h > 0. \) Prove that the method is globally convergent, and the global rate of convergence is second order, assuming that \( y \in C^3[a, b]. \)

3. Consider the IVP \( y'(x) = y(x), \ x \in [0, 1], y(0) = 1. \) Apply the following three methods to obtain approximations of \( y(1) = e: \)

(a) Taylor’s Method:

\[
\eta_i+1 = \eta_i + h \left\{ f(x_i, \eta_i) + \frac{h}{2} [f_x(x_i, \eta_i) + f_y(x_i, \eta_i)f(x_i, \eta_i)] \right\},
\]

(b) Heun’s Method:

\[
\eta_i+1 = \eta_i + \frac{h}{2} \left\{ f(x_i, \eta_i) + f(x_{i+1}, \eta_i + hf(x_i, \eta_i)) \right\},
\]

(c) Modified Euler’s Method:

\[
\eta_i+1 = \eta_i + h \left\{ f\left(x_i + \frac{h}{2}, \eta_i + \frac{h}{2} f(x_i, \eta_i)\right) \right\}.
\]
4. Show that the 2-step (implicit) Adams-Moulton method

\[ \eta_{i+2} - \eta_{i+1} = h \left[ \frac{5}{12} f(x_{i+2}, \eta_{i+2}) + \frac{8}{12} f(x_{i+1}, \eta_{i+1}) - \frac{1}{12} f(x_i, \eta_i) \right] \]

is consistent using the simple-to-check conditions \( \psi(1) = 0 \) and \( \psi'(1) - \chi(1) = 0 \). Prove that, in fact, the method is consistent and third-order using the difference

\[ \frac{\psi(\mu)}{\ln(\mu)} - \chi(\mu). \]

Show that the method satisfies the root condition.

5. Show that the 3-step (implicit) Adams-Moulton method

\[ \eta_{i+3} - \eta_{i+2} = h \left[ \frac{9}{24} f(x_{i+3}, \eta_{i+3}) + \frac{19}{24} f(x_{i+2}, \eta_{i+2}) - \frac{5}{24} f(x_{i+1}, \eta_{i+1}) + \frac{1}{24} f(x_i, \eta_i) \right] \]

is consistent using the simple-to-check conditions \( \psi(1) = 0 \) and \( \psi'(1) - \chi(1) = 0 \). Prove that, in fact, the method is consistent and fourth-order using the difference

\[ \frac{\psi(\mu)}{\ln(\mu)} - \chi(\mu). \]

Show that the method satisfies the root condition.

6. Find all of the values of \( \alpha \) and \( \beta \) so that the 3-step method

\[ \eta_{i+3} + \alpha(\eta_{i+2} - \eta_{i+1}) - \eta_i = h\beta [f(x_{i+2}, \eta_{i+2}) + f(x_{i+1}, \eta_{i+1})] \]

has local order of accuracy 4. Show that the resulting method does not satisfy the root condition and, therefore, is not convergent (and, thus, not at all useful).

7. Show (directly) that the the BDF3 scheme

\[ \eta_{i+3} = \frac{18}{11} \eta_{i+2} + \frac{9}{11} \eta_{i+1} - \frac{2}{11} \eta_i = \frac{6}{11} h f(x_{i+3}, \eta_{i+3}) \]

satisfies the root condition. Show (directly) that it is also consistent, and so conclude that it is a convergent method.

**Hint:** One of the roots is \( w = 1 \).
8. Consider the scheme

\[ \eta_{i+2} - \eta_i = \frac{1}{3} h \left( f(x_{i+2}, \eta_{i+2}) + 4 f(x_{i+1}, \eta_{i+1}) + f(x_i, \eta_i) \right). \]

Show that it is of fourth order, and it obeys the root condition.