1. (20 points): A nitric acid solution flows at a constant rate of 6 L/min into a large tank that initially held 200 L of 0.5% nitric acid solution. The solution inside the tank is kept well stirred and flows out of the tank at a rate of 6 L/min. If the solution entering the tank is 20% nitric acid, determine the volume of nitric acid in the tank after $t$ minutes.

Solution: (See problem 3, page 98.) Let $V(t)$ be the volume of solution (water and nitric acid) measured in liters after $t$ minutes. Let $x(t)$ be the volume of nitric acid measured in liters after $t$ minutes, and let $c(t)$ be the concentration (by volume) of nitric acid in solution after $t$ minutes.

The volume of solution $V(t)$ doesn’t change over time since the inflow and outflow of solution is equal. Thus $V = 200$ L. The concentration of nitric acid $c(t)$ is

$$c(t) = \frac{x(t)}{V(t)} = \frac{x(t)}{200}.$$

We model this problem as

$$\frac{dx}{dt} = I(t) - O(t),$$

where $I(t)$ is the input rate of nitric acid and $O(t)$ is the output rate of nitric acid, both measured in liters of nitric acid per minute. The input rate is

$$I(t) = \frac{6\ \text{L}_{\text{sol.}}}{1\ \text{min}} \cdot \frac{20\ \text{L}_{\text{nit.}}}{100\ \text{L}_{\text{sol.}}} = \frac{120\ \text{L}_{\text{nit.}}}{100\ \text{min}} = 1.2\ \text{L}_{\text{nit.}}/\text{min}.$$

The output rate is

$$O(t) = (6\ \text{L}_{\text{sol.}}/\text{min})c(t) = \frac{6\ \text{L}_{\text{sol.}}}{1\ \text{min}} \cdot \frac{x(t)\ \text{L}_{\text{nit.}}}{200\ \text{L}_{\text{sol.}}} = \frac{3x(t)\ \text{L}_{\text{nit.}}}{100\ \text{min}} = 0.03\ x(t)\ \text{L}_{\text{nit.}}/\text{min}.$$

The equation is then

$$\frac{dx}{dt} = 1.2 - 0.03x,$$
or \[
\frac{dx}{dt} + 0.03x = 1.2, \tag{1}
\]
which is a linear equation. The initial condition condition is found in the following way:
\[
c(0) = 0.5\% = \frac{5 \text{ L}_{\text{init.}}}{1000 \text{ L}_{\text{sol.}}} = \frac{x(0) \text{ L}_{\text{init.}}}{200 \text{ L}_{\text{sol.}}}.
\]
Thus \(x(0) = 1\).

In Eq. (1) we let \(P(t) = 0.03\) and \(Q(t) = 1.2\). The integrating factor for Eq. (1) is
\[
\mu(t) = \exp \left\{ \int P(t) \, dt \right\} = \exp \left\{ 0.03 \int dt \right\} = e^{0.03t}.
\]
The solution is
\[
x(t) = \frac{1}{\mu(t)} \left\{ \int \mu(t)Q(t) \, dt + C \right\}
= Ce^{-0.03t} + 1.2e^{-0.03t} \int e^{0.03t} \, dt
= Ce^{-0.03t} + \frac{1.2}{0.03} e^{-0.03t} e^{0.03t}
= Ce^{-0.03t} + \frac{1.2}{0.03}
= Ce^{-0.03t} + 40.
\]
The constant is found using \(x(t) = 1\):
\[
x(0) = Ce^{-0.03(0)} + 40 = C + 40 = 1.
\]
Thus \(C = -39\), and the solution is
\[
x(t) = -39e^{-0.03t} + 40.
\]
2. (20 points): Solve the following equation:

\[(x^2y^3 + y)\,dx + x\,dy = 0.\]

**Solution:** (See problem 25, page 79.) This is a Bernoulli type equation with \(n = 3\), \(P(x) = 1/x\), and \(Q(x) = -x\):

\[\frac{dy}{dx} + \frac{1}{x}y = -xy^3.\]

Set \(v = y^{1-n} = y^{1-3} = y^{-2}\). Then the equation becomes

\[\frac{1}{1-n} \frac{dv}{dx} + P(x)v = Q(x),\]

or

\[\frac{1}{-2} \frac{dv}{dx} + \frac{1}{x}v = -x,\]

or

\[\frac{dv}{dx} + \frac{-2}{x}v = 2x.\]  \hspace{1cm} (2)

Equation (2) is a linear equation, which we solve using an integrating factor:

\[
\begin{align*}
\mu(x) &= \exp \left\{ \int P(x) \, dx \right\} = \exp \left\{ \int -2/x \, dx \right\} = \exp \left\{ -2 \int 1/x \, dx \right\} \\
&= \exp \{-2 \ln |x|\} = \exp \{\ln(1/x^2)\} = 1/x^2.
\end{align*}
\]

The solution is

\[
v(x) = \frac{1}{\mu(x)} \left\{ \int \mu(x)Q(x) \, dx + C \right\}
\]

\[= \frac{1}{x^2} \int 2x/x^2 \, dx
\]

\[= Cx^2 + 2x^2 \int 1/x \, dx
\]

\[= Cx^2 + 2x^2 \ln |x|.
\]

Thus

\[y^{-2} = Cx^2 + 2x^2 \ln |x|.
\]
3. (20 points): Solve the following equation:

\[(2xy^3 + 1) \, dx + (3x^2y^2 - 1/y) \, dy = 0.\]

Solution: (See problem 12, page 71.) This equation is exact. Set \( M(x, y) = 2xy^3 + 1, N(x, y) = 3x^2y^2 - 1/y. \) Then

\[
\frac{\partial M}{\partial y} = 6xy^2 = \frac{\partial N}{\partial x}.
\]

The solution is of the form \( F(x, y) = C, \) where

\[
\frac{\partial F(x, y)}{\partial x} = M(x, y)
\]

and

\[
\frac{\partial F(x, y)}{\partial y} = N(x, y).
\]

Thus

\[
F(x, y) = \int M(x, y) \, dx + g(y) = \int (2xy^3 + 1) \, dx + g(y)
\]

\[= x^2y^3 + x + g(y).\]

And,

\[3x^2y^2 - 1/y = N(x, y) = \frac{\partial F(x, y)}{\partial y} = \frac{\partial}{\partial y} (x^2y^3 + x) + g'(y) = 3x^2y^2 + g'(y).\]

Thus

\[g'(y) = -1/y,
\]

or

\[g(y) = -\ln |y|.
\]

Hence, the solution is

\[F(x, y) = x^2y^3 + x - \ln |y| = C.\]
4. **(20 points):** If the following equation is exact, solve it directly. If not, make it exact using a special integrating factor, then solve it.

\[(y^2 + 2xy) \, dx - x^2 \, dy = 0.\]

**Solution:** (See problem 11, page 71.) Note immediately that \(y \equiv 0\) is a solution.

The equation is not exact. Set \(M(x, y) = y^2 + 2xy\) and \(N(x, y) = -x^2\). Then

\[\frac{\partial M}{\partial y} = 2y + 2x \neq -2x = -\frac{\partial N}{\partial x}.\]

Consider

\[\frac{\partial N/\partial x - \partial M/\partial y}{M} = \frac{-2x - 2y - 2x}{y^2 + 2xy} = \frac{-2y - 4x}{y^2 + 2xy} = \frac{(-2)(y + 2x)}{(y)(y + 2x)} = -\frac{2}{y}.\]

Since this is only a function of \(y\), we may construct an integrating factor \(\mu(y)\) that makes the equation exact:

\[\mu(y) = \exp \left\{ \int -2/y \, dy \right\} = \exp \left\{ -2 \int 1/y \, dy \right\} = \exp \left\{ -2 \ln |y| \right\} = \exp \left\{ \ln(1/y^2) \right\} = y^{-2}.\]

Thus the equation

\[y^{-2} \left( y^2 + 2xy \right) \, dx - y^{-2}x^2 \, dy = 0,\]

or

\[(1 + 2x/y) \, dx - (x/y)^2 \, dy = 0\]

is exact.

Set \(\bar{M} = 1 + 2x/y\) and \(\bar{N} = -(x/y)^2\). Then

\[\frac{\partial \bar{M}}{\partial y} = \frac{-2x}{y^2} = \frac{\partial \bar{N}}{\partial x}.\]

Thus the solution is of the form \(F(x, y) = C\), where

\[\frac{\partial F(x, y)}{\partial x} = \bar{M}(x, y)\]

and

\[\frac{\partial F(x, y)}{\partial y} = \bar{N}(x, y).\]

Thus

\[F(x, y) = \int \bar{M}(x, y) \, dx + g(y) = \int (1 + 2x/y) \, dx + g(y) = x + x^2/y + g(y).\]
And,

\[-(x/y)^2 = \tilde{N}(x, y) = \frac{\partial F(x, y)}{\partial y} = \frac{\partial}{\partial y} (x + x^2/y) + g'(y) = -(x/y)^2 + g'(y).\]

Thus

\[g'(y) = 0\]

and

\[g(y) = 0.\]

Hence, the solution is

\[F(x, y) = x + x^2/y = C.\]

and, recall, \(y \equiv 0\) is also a solution.
5. (20 points): Determine the equation for the displacement $y(t)$ for a mass spring system where $m = 36$ kg, $b = 12$ kg/sec (equivalently, $b = 12$ N · sec/m), $k = 37$ kg/sec$^2$, and $F_{ext} = 0$. Suppose the initial displacement is $y(0) = 1$ m, and the initial velocity is $y'(0) = 0$ m/sec.

Solution: (See example 3, page 173.) The model for a damped mass-spring oscillator is

$$my''(t) + by'(t) + ky(t) = F_{ext}.$$ 

Thus, plugging in the numbers,

$$36y''(t) + 12y'(t) + 37y(t) = 0.$$ 

The auxiliary equation is

$$36r^2 + 12r + 37 = 0.$$ 

The solutions to the auxiliary equation are

$$r = \frac{-12 \pm \sqrt{12^2 - 4(36)(37)}}{2(36)} = \frac{-12 \pm \sqrt{-5184}}{72} = \frac{-12 \pm 72i}{72} = -1/6 \pm i.$$ 

Thus the solution is

$$y(t) = e^{-t/6} (A \cos(t) + B \sin(t)).$$ 

The initial conditions are $y(0) = 1$ and $y'(0) = 0$. Hence

$$1 = y(0) = e^{-0/6} (A \cos(0) + B \sin(0)) = A,$$

which implies $A = 1$. Also

$$0 = y'(0) = (-1/6)e^{-0/6} (A \cos(0) + B \sin(0)) + e^{-0/6} (-A \sin(0) + B \cos(0))$$

$$= -A/6 + B = -1/6 + B.$$ 

Hence $B = 1/6$. The complete solution is

$$y(t) = e^{-t/6} (\cos(t) + (1/6) \sin(t)).$$
6. **(20 points):** Solve the following equation:

\[
\frac{dy}{dx} = \frac{y^2 + x \sqrt{x^2 + y^2}}{xy}.
\]

**Solution:** (See problem 13, page 79.) This equation is homogeneous. Set \( v = y/x \), then

\[
\frac{dy}{dx} = \frac{x^2}{x^2} \cdot \frac{y^2 + x \sqrt{x^2 + y^2}}{xy} = \frac{(y/x)^2 + (1/x) \sqrt{x^2 + y^2}}{y/x}
\]

\[
= \frac{(y/x)^2 + \sqrt{(1/x^2)} (x^2 + y^2)}{y/x}
\]

\[
= \frac{(y/x)^2 + \sqrt{1 + (y/x)^2}}{y/x}
\]

\[
= \frac{v^2 + \sqrt{1 + v^2}}{v} = G(v).
\]

The transformed equation is

\[
v + x \frac{dv}{dx} = G(v),
\]

or

\[
\frac{dv}{dx} = \frac{G(v) - v}{x},
\]

which is separable. Simplifying, we have

\[
\frac{dv}{dx} = \frac{v^2 + \sqrt{1 + v^2} - v^2}{xv} = \frac{\sqrt{1 + v^2}}{xv}.
\]

The solution may be found by integrating:

\[
\int \frac{v \ dv}{\sqrt{1 + v^2}} = \int \frac{dx}{x} = \ln |x| + C
\]

\[
\int \frac{v \ dv}{\sqrt{1 + v^2}} = \frac{1}{2} \int u^{-1/2} \ du = u^{1/2} = \sqrt{1 + v^2}.
\]

Thus, the solution is

\[
\sqrt{1 + v^2} = \ln |x| + C,
\]

or

\[
\sqrt{1 + (y/x)^2} = \ln |x| + C.
\]
7. **(20 points):** Solve the following equation:

\[
\frac{dy}{dx} - x^3(1 - y) = 0, \quad y(0) = 3.
\]

**Solution:** (See problem 17, page 46.) This equation is separable:

\[
\frac{dy}{dx} = x^3(1 - y).
\]

The solution may be found by integrating

\[
\int \frac{dy}{1 - y} = \int x^3 \, dx = \frac{x^4}{4} + C.
\]

\[
\int \frac{dy}{1 - y} = \int \frac{-du}{u} = -\ln |u| = -\ln |1 - y|.
\]

Thus

\[-\ln |1 - y| = \frac{x^4}{4} + C,
\]

or

\[
\ln |1 - y| = -\frac{x^4}{4} + C_1.
\]

Hence

\[
\exp \{\ln |1 - y|\} = \exp \left\{-\frac{x^4}{4} + C_1\right\},
\]

or, equivalently,

\[
|1 - y| = C_2 e^{-x^4/4}.
\]

The sign from the absolute values may be absorbed in the constant, so that

\[
1 - y = C_3 e^{-x^4/4},
\]

thus

\[
y(x) = 1 - C_3 e^{-x^4/4}.
\]

Using the initial condition \(y(0) = 3\), we have

\[
3 = y(0) = 1 - C_3 e^{-(0)^4/4} = 1 - C_3,
\]

which implies that \(C_3 = -2\). The solution is, finally,

\[
y(x) = 1 + 2e^{-x^4/4}.
\]