1. Suppose that $A \in \mathbb{R}^{n \times n}$ is SPD. (a) Show that $\|x\|_A = \sqrt{x^T A x}$ defines a matrix norm. (b) Let the eigenvalues of $A$ be ordered so that $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Show that

$$\sqrt{\lambda_1} \|x\|_2 \leq \|x\|_A \leq \sqrt{\lambda_n} \|x\|_2$$

for any $x \in \mathbb{R}^n$.

2. Suppose that $A \in \mathbb{R}^{n \times n}$ is SPD. Prove that $x^* \in \mathbb{R}^n$ solves $A x = b$ iff $x^*$ minimizes the quadratic function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{2} x^T A x - x^T b .$$

3. Define $A \in \mathbb{R}^{(n-1) \times (n-1)}$ via

$$A = \begin{bmatrix}
2 & -1 & 0 & \cdots & 0 \\
-1 & 2 & \ddots & \vdots \\
0 & \ddots & \ddots & -1 & 0 \\
\vdots & & -1 & 2 & -1 \\
0 & \cdots & 0 & -1 & 2
\end{bmatrix} .$$

(If you like, take $N + 1 = n$.) Let $h = 1/n$, where $n$ is positive integer. Consider the set

$$S = \left\{ w^{(1)}, w^{(2)}, \ldots, w^{(n-1)} \right\} ,$$

where the $i^{th}$ component of $w^{(k)}$ is defined via

$$w_i^{(k)} = \sin \left( k \pi i h \right) .$$

Show that $S$ is a set of eigenvectors of $A$, and compute the eigenvalues of $A$. Are the eigenvectors orthogonal? Conclude that $A$ is a positive definite matrix.

4. Compute $\kappa_2(A)$ for the previous matrix.

5. Let $A$ again be defined as above. By calculating $x^T A x$, show directly that $A$ is positive definite.

**Hint:** Use

$$2x_i^2 - 2x_i x_{i+1} + 2x_{i+1}^2 = x_i^2 + (x_i - x_{i+1})^2 + x_{i+1}^2 .$$