1. Suppose that $A \in \mathbb{R}^{n \times n}$ is SPD and define the quadratic function $f : \mathbb{R}^n \to \mathbb{R}$ via

$$f(x) = \frac{1}{2} x^T A x - x^T b.$$ 

Define $r_{n-1} := b - A x_{n-1}$. Prove that

$$\alpha_n := \arg \min_{\alpha \in \mathbb{R}} f(x_{n-1} + \alpha r_{n-1}) = \frac{r_{n-1}^T r_{n-1}}{r_{n-1}^T A r_{n-1}}.$$ 

2. Define $A \in \mathbb{R}^{m \times m}$ via

$$A = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & \ddots & \vdots \\ 0 & \ddots & \ddots & -1 & 0 \\ \vdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix}.$$ 

Estimate the rate of convergence of the CG iteration using formula (38.10).

3. Calculate $\rho(T_J)$ and $\rho(T_{GS})$ for the matrix in the previous problem. Also, calculate the optimal value for $\omega$ in the SOR method.

4. Suppose that $A \in \mathbb{R}^{n \times n}$ is SPD. Prove that the Gauss-Seidel method converges. Do this in the following steps:

(a) Show that $A = D - L - L^T$ where $D$ is diagonal with $d_{ii} > 0$ for $i = 1, \ldots, n$, and $L$ is lower triangular. Show that $D - L$ is invertible.

(b) Let $T_{GS} = (D - L)^{-1} L^T$ and $P = A - T_{GS}^T A T_{GS}$. Prove that $P$ is symmetric.

(c) Show that $T_{GS} = I - (D - L)^{-1} A$.

(d) Let $Q = (D - L)^{-1} A$. Prove that $T_{GS} = I - Q$ and

$$P = Q^T [AQ^{-1} - A + Q^{-T} A] Q.$$ 

(e) Show that $P = Q^T D Q$ and that $P$ is SPD.

(f) Let $\lambda$ be an eigenvalue of $T_{GS}$ with eigenvector $x$. Show that $x^T P x > 0$ implies that $|\lambda| < 1$.

(g) Show that $T_{GS}$ is convergent.