Final Exam, Math 121 A
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NAME: Solutions

ID:

Instructions: Put away any electronic devices, papers, notes, and books. Use only a pencil and eraser (or a pen if you are daring). Write neatly and only on the paper provided. Use the back of the page if necessary.

1. (20 points): Let $V$ and $W$ be vector spaces, let $T : V \to W$ be linear and let \( \{w_1, w_2, \ldots, w_k\} \) be a finite, linearly independent subset of $R(T)$. Prove that if $S = \{v_1, v_2, \ldots, v_k\}$ is chosen so that $T(v_i) = w_i$ for $i = 1, 2, \ldots, k$, then $S$ is linearly independent.

Solution: Consider an arbitrary representation of zero as a linear combination of vectors from $S$:

\[
\sum_{i=1}^{k} a_i v_i = 0 \in V. \tag{1}
\]

Note that $0 \in V$ always gets mapped to $0 \in W$, provided $T$ is linear. Thus

\[
0 \in W = T(0) = T \left( \sum_{i=1}^{k} a_i v_i \right). \tag{2}
\]

By the linearity of $T$ we have

\[
0 \in W = T \left( \sum_{i=1}^{k} a_i v_i \right) = \sum_{i=1}^{n} a_i T(v_i) = \sum_{i=1}^{n} a_i w_i. \tag{3}
\]

Since \( \{w_1, w_2, \ldots, w_k\} \) is linearly independent, the only way for the last equality to hold is if $a_i = 0$ for $i = 1, \ldots, n$. This implies that Eq. (1) is the trivial representation of zero. Hence $S$ is linearly independent.///
2. **(20 points):** Let \( V \) and \( W \) be vector spaces and let \( T : V \to W \) be linear. If \( V_0 \) is a vector subspace of \( V \) show that \( T(V_0) = \{ w \in R(T) \mid T(v) = w \text{ for some } v \in V_0 \} \) is a vector subspace of \( W \).

**Solution:** We must show that (a) \( 0_W \in T(V_0) \), (b) \( w_1 + w_2 \in T(V_0) \), for all \( w_1, w_2 \in T(V_0) \), and (c) \( cw_1 \in T(V_0) \), for all \( w_1 \in T(V_0) \) and \( c \in F \).

(a) Since \( T \) is linear \( T(0_V) = 0_W \). It must be true that \( 0_V \in V_0 \), because \( V_0 \) is a vector subspace of \( V \). Thus \( 0_W \in T(V_0) \), because there is some element \( v \in V_0 \), namely \( v = 0_V \), such that \( T(v) = 0_W \).

(b) Let \( w_1, w_2 \in T(V_0) \). Then there exist \( v_1, v_2 \in V_0 \) such that \( T(v_1) = w_1 \) and \( T(v_2) = w_2 \). By the linearity of \( T \), \( w_1 + w_2 = T(v_1 + v_2) \). Now, it must be true that \( v_1 + v_2 \in V_0 \), because \( v_1, v_2 \in V_0 \) and \( V_0 \) is a vector subspace of \( V \). Thus \( w_1 + w_2 \in T(V_0) \), because there is some element \( v \in V_0 \), namely \( v = v_1 + v_2 \), such that \( T(v) = w_1 + w_2 \).

(c) Let \( w_1 \in T(V_0) \). Then there exists some \( v_1 \in V_0 \) such that \( T(v_1) = w_1 \). Let \( c \in F \). Then \( cv_1 \in V_0 \), because \( V_0 \) is a vector subspace of \( V \). By the linearity of \( T \), \( cw_1 = T(cv_1) \). Thus \( cw_1 \in T(V_0) \), because there is some element \( v \in V_0 \), namely \( v = cv_1 \), such that \( T(v) = cw_1 \).
3. **(20 points):** Let $V$ and $W$ be finite-dimensional vector spaces and $T : V \rightarrow W$ be an isomorphism from $V$ onto $W$. Let $V_0$ be a vector subspace of $V$. Prove that $\dim(V_0) = \dim(T(V_0))$.

**Solution:** Since $T : V \rightarrow W$ is an isomorphism, it is one-to-one and onto. Since $V_0$ is a vector subspace of $V$, $T(V_0)$ is a vector subspace of $W$, by problem 2. Moreover, both $V_0$ and $T(V_0)$ must be finite dimensional. Now consider the restricted mapping $T : V_0 \rightarrow T(V_0)$. The restricted mapping is still linear, is onto by definition, and is also one-to-one. (If it was not one-to-one, then the full mapping from $V$ onto $W$ would not be one-to-one.) Thus the restricted mapping is an isomorphism from $V_0$ onto $T(V_0)$. By **Theorem 2.19** $\dim(V_0) = \dim(T(V_0))$.///
4. (30 points): Let $T : V \rightarrow W$ be a linear transformation from an $n$-dimensional vector space $V$ to an $m$-dimensional vector space $W$. Let $\beta$ and $\gamma$ be ordered bases for $V$ and $W$, respectively. Prove that $\text{rank}(T) = \text{rank}(L_A)$ and $\text{nullity}(T) = \text{nullity}(L_A)$, where $A = [T]_{\beta}^{\gamma}$.

Hint: Use the results of the previous two problems, the appropriate diagram, and the fact that $[T(v)]_{\gamma} = [T]_{\beta}^{\gamma}[v]_{\beta}$. Don’t use Theorem 3.3: $\text{rank}(T) = \text{rank}([T]_{\beta}^{\gamma})$.

Solution: Consider the figure above, where $\phi_{\beta} : V \rightarrow F^n$ is the linear transformation defined by $\phi_{\beta}(v) = [v]_{\beta}$, and $\phi_{\gamma} : V \rightarrow F^m$ is the linear transformation defined by $\phi_{\gamma}(w) = [w]_{\gamma}$. These transformations are isomorphisms between their respective spaces. This proof has 3 parts.

(1) We will first show that $\phi_{\beta}(N(T)) = N(L_A): (\subseteq) Let x \in \phi_{\beta}(N(T)). Since \phi_{\beta}$ is an isomorphism between $V$ and $F^n$ there exists a unique $v \in N(T)$ such that $\phi_{\beta}(v) = [v]_{\beta} = x$. Now since $v \in N(T)$, $T(v) = 0_W$. Using the fact from the hint, we have

$$\theta_{F^n} = [\theta_W]_{\gamma} = [T(v)]_{\gamma} = [T]_{\beta}^{\gamma}[v]_{\beta} = Ax.$$ 

Thus $x \in N(L_A).$/

(\supseteq) Now let $x \in N(L_A)$. Let $v \in V$ be the unique vector such that $\phi_{\beta}(v) = [v]_{\beta} = x$. Then we have

$$\theta_{F^n} = Ax = [T]_{\beta}^{\gamma}[v]_{\beta} = [T(v)]_{\gamma} = \phi_{\gamma}(T(v)).$$

Since $\phi_{\gamma}$ is an isomorphism, $T(v) = \theta_W$. Thus $v \in N(T)$, which implies that $x \in \phi_{\beta}(N(T)).$/

(2) By problem 3, and what we’ve just proven

$$\text{nullity}(T) = \dim(N(T)) = \dim(\phi_{\beta}(N(T))) = \dim(N(L_A)) = \text{nullity}(L_A).$$
(3) Finally, by the **Dimension Theorem**

\[
\begin{align*}
    n &= \dim(V) = \text{nullity}(T) + \text{rank}(T), \\
    n &= \dim(F^n) = \text{nullity}(L_A) + \text{rank}(L_A).
\end{align*}
\]

Thus we also have \(\text{rank}(T) = \text{rank}(L_A)\.///\)
5. (20 points): Suppose that $A, B \in M_{n \times n}$ and let $M \in M_{2n \times 2n}$ be defined by

$$M = \begin{pmatrix} A & B \\ O & I_n \end{pmatrix},$$

where $I_n$ is the $n \times n$ identity and $O$ is the $n \times n$ zero matrix. Show that $\det(M) = \det(A)$.

**Solution:** Let $M^k$ denote the $(2n - k) \times (2n - k)$ square matrix resulting from deleting the last $k$ rows and last $k$ columns of $M$. Let $B^k$ be the $n \times (n - k)$ matrix resulting from deleting only the last $k$ columns of $B$, and let $O^k$ be the $(n - k) \times n$ matrix resulting from deleting only the last $k$ rows of the $n \times n$ zero matrix $O$. Clearly, $M^n = A$, and for $1 \leq k \leq n - 1$,

$$M^k = \begin{pmatrix} A & B^k \\ O^k & I_{n-k} \end{pmatrix},$$

By definition, expanding about the last row of $M$,

$$\det(M) = \sum_{j=1}^{2n} (-1)^{2n+j} M_{2n,j} \det(\tilde{M}_{2n,j}),$$

where $\tilde{M}_{2n,j}$ is the $(2n - 1) \times (2n - 1)$ square matrix resulting from deleting row $2n$ and column $j$. The only nonzero element of row $2n$ of $M$ is the last one, i.e., the one for which $j = 2n$, and equals 1. Thus

$$\det(M) = (-1)^{2n+2n} M_{2n,2n} \det(\tilde{M}_{2n,2n})$$
$$= \det(\tilde{M}_{2n,2n})$$
$$= \det(M^1).$$

The determinant for the matrix $M^k$, for $1 \leq k \leq n - 1$, may be calculated similarly. Expanding about the last row of $M^k$, we have

$$\det(M^k) = \sum_{j=1}^{2n-k} (-1)^{2n-k+j} M_{2n-k,j}^k \det(\tilde{M}_{2n-k,j}^k)$$
$$= (-1)^{2n-k+2n-k} M_{2n-k,2n-k}^k \det(\tilde{M}_{2n-k,2n-k}^k)$$
$$= \det(M_{2n-k,2n-k}^k)$$
$$= \det(M^{k+1}).$$

Thus we have

$$\det(M) = \det(M^1) = \det(M^2) = \cdots = \det(M^n) = \det(A).$$

Which shows $\det(M) = \det(A)$. ///
6. **(20 points):** Find a basis for the null space of the linear transformation $L_A$, where

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & -1 & 1 & -1 \\ -2 & 1 & -1 & 1 \end{pmatrix}.$$

**Solution:** We will put $(A|\theta)$ into reduced row echelon form.

$$(A|\theta) = \begin{pmatrix} 0 & 1 & 0 & 2 & 0 \\ 1 & -1 & 1 & -1 & 0 \\ -2 & 1 & -1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ -2 & 1 & -1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & -1 & 1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

The last augmented matrix represents the linear system of equations

$$x_1 = 0, \quad x_2 = -2x_4, \quad x_3 = -x_4.$$

Setting $x_4 = t$, we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -2t \\ -t \\ t \end{pmatrix} = t \begin{pmatrix} 0 \\ -2 \\ -1 \\ 1 \end{pmatrix}.$$

Thus $\{(0, -2, -1, 1)^t\}$ is a basis for the null space of $L_A$.///
7. **(30 points):** Let $T : V \to W$ be a linear transformation from an $n$-dimensional vector space $V$ to another $n$-dimensional vector space $W$. Let $\beta$ be an ordered bases for $V$. Prove that $T$ is an isomorphism if and only if $T(\beta)$ is a basis for $W$.

**Solution:** ($\Rightarrow$) Suppose that $T : V \to W$ is an isomorphism. Let $a_i \in F$, $i = 1, \ldots, n$, be arbitrary, set $\beta = \{v_i\}_{i=1}^n$, and consider

$$0_W = \sum_{i=1}^n a_i T(v_i) = T\left(\sum_{i=1}^n a_i v_i\right) \text{ (by linearity of } T).$$

Set $v = \sum_{i=1}^n a_i v_i$, then $v \in N(T)$. But, since $T$ is one-to-one and onto, $N(T) = \{\theta_W\}$, which implies that $v = \theta_V$. Thus we have

$$0_V = \sum_{i=1}^n a_i v_i.$$

Since $\beta = \{v_i\}_{i=1}^n$ is a basis for $V$, this implies that $a_i = 0$, for $i = 1, \ldots, n$. Thus $T(\beta)$ is a basis for $W$.

($\Leftarrow$) Suppose that $T(\beta)$ is a basis for $W$. We’ll show that $T$ is onto. Let $w \in W$ be arbitrary. Since $T(\beta)$ is a basis for $W$, there exist unique scalars $a_1, \ldots, a_n \in F$ such that

$$w = \sum_{i=1}^n a_i T(v_i) = T\left(\sum_{i=1}^n a_i v_i\right).$$

Set $v = \sum_{i=1}^n a_i v_i$. Since $\beta = \{v_i\}_{i=1}^n$ is a basis for $V$, this implies that $v \in V$. Thus $w \in R(T)$, which implies that $W \subseteq R(T)$. Since it is always true that $R(T) \subseteq W$, it must be that $W = R(T)$. Hence $T$ is onto. By **Theorem 2.5**, $T$ is one-to-one. This proves that $T$ is an isomorphism.///