Cyclic vectors in the Drury-Arveson space.

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joint work with Carl Sundberg

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\[ d \in \mathbb{N}, \quad B_d = \{ z \in \mathbb{C}^d : |z| < 1 \} \]

If \( d = 1 \), then \( \mathcal{D} = B_1 \)
$d \in \mathbb{N}, \quad \mathbb{B}_d = \{z \in \mathbb{C}^d : |z| < 1\}$

If $d = 1$, then $D = \mathbb{B}_1$

$z = (z_1, ..., z_d), \ w = (w_1, ..., w_d), \ \lambda \in \mathbb{C}$

$$\langle z, w \rangle = z_1 \overline{w}_1 + ... + z_d \overline{w}_d$$

$$\lambda z = (\lambda z_1, ..., \lambda z_d)$$

$M_z = (M_{z_1}, ..., M_{z_d})$
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$$\langle z, w \rangle = z_1 \overline{w}_1 + ... + z_d \overline{w}_d$$

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Multiindex notation: If $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}_0^d$, then

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_d^{\alpha_d}$$

$$\alpha! = \alpha_1! \cdots \alpha_d! \quad |\alpha| = \alpha_1 + ... + \alpha_d$$
The **Drury-Arveson space** $H^2_d$ is defined by the reproducing kernel

$$k_w(z) = \frac{1}{1 - \langle z, w \rangle}, \quad z, w \in \mathbb{B}_d$$
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Other spaces of analytic functions in $\mathbb{B}_d$:

- $H^2(\partial \mathbb{B}_d) - k_w(z) = \left(\frac{1}{1 - \langle z, w \rangle}\right)^d$
- $L^2_a(\mathbb{B}_d) - k_w(z) = \left(\frac{1}{1 - \langle z, w \rangle}\right)^{d+1}$
Interesting fact about \((M_z, H^2_d)\):

Each \(M_{z_i}\) is subnormal, but \((M_z, H^2_d)\) is not jointly subnormal.

\[
\begin{align*}
\triangleright & \quad (M_{z_1}, H^2_d) \text{ u.e. } (M_z, H^2(D)) \oplus ((M_z, L^2(D)) \otimes I_d) \oplus \ldots \\
\triangleright & \quad M(H^2_d) \nsubseteq H^\infty(D)
\end{align*}
\]
$H^2_d$ has been proposed as a possible analogue of $H^2(\mathbb{D})$, useful for the study of commuting row contractions $T = (T_1, ..., T_d)$

$$
\| (T_1, \ldots, T_d) \begin{pmatrix} x_1 \\ x_d \end{pmatrix} \|_2^2 = \| \sum_j T_j x_j \|_2^2 \leq \| x_1 \|_2^2 + \cdots + \| x_d \|_2^2
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$$\| (T_1, \ldots, T_d) \binom{x_1}{x_d} \|^2 = \| \sum_j T_j x_j \|^2 \leq \| x_1 \|^2 + \cdots + \| x_d \|^2$$

Evidence and successes:

- dilation theorem (Drury, Mueller-Vasilescu, Arveson)
  \[ \exists M \in \text{Lat}(M_z \otimes I) \oplus U \]
  \[ T_i = P_M \perp ((M_{z_i} \otimes I) \oplus U_i) | M \perp, \quad i = 1, \ldots, d \]

- analogue of von Neumann's inequality (Drury)

- commutant lifting theorem (Ball-Trent-Vinnikov, Davidson-Le)

- Corona theorem for $M(\mathcal{H}^2_d)$ (Costea-Sawyer-Wick)
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- analogue of von Neumann’s inequality (Drury)
  \[ \| p(T_1, \ldots, T_d) \| \leq \| p \|_{M(H^2_d)} \ \forall \ \text{polys } p \]
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▷ analogue of von Neumann’s inequality (Drury)

$$\| p(T_1, ..., T_d) \| \leq \| p \|_{M(H^2_d)} \quad \forall \text{ polys } p$$

▷ commutant lifting theorem (Ball-Trent-Vinnikov, Davidson-Le)

▷ Corona theorem for $M(H^2_d)$ (Costea-Sawyer-Wick)
Much is based on the fact that $k_w(z) = \frac{1}{1-\langle z, w \rangle}$ is a complete Nevanlinna-Pick kernel.
Much is based on the fact that $k_w(z) = \frac{1}{1 - \langle z, w \rangle}$ is a complete Nevanlinna-Pick kernel. Ideally a function theory of $H^2_d$ would take advantage of results about Besov spaces of the ball and the special form of $k_w$. 
Thm (Gleason-R-Sundberg)

If $\mathcal{M} \in \text{Lat}(M_z, H^2_d)$, if $T_i = P_{\mathcal{M}^\perp} M_{z_i} | M^\perp$ for all $i$, then

$$\sigma(T) \cap \mathbb{B}_d = Z(\mathcal{M}) = \{z \in \mathbb{B}_d : f(z) = 0 \ \forall \ f \in \mathcal{M}\}.$$

**Remark:** This is false for some invariant subspaces of $H^2(\partial B_d)$, $d > 1$. 
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What about $\sigma(T) \cap \partial B^d$?
invariant subspaces - inner sequences

Thm (McCullough-Trent, Arveson)
If $\mathcal{M} \in \text{Lat}(\mathcal{M}_z, H^2_d)$, then there are $\varphi_1, \varphi_2, \ldots \in \mathcal{M}(H^2_d)$ such that

$$P_{\mathcal{M}} = \sum_{n} M_{\varphi_n} M_{\varphi_n}^*$$

$$\mathcal{M} = \left\{ \sum_{n} \varphi_n f_n : f_n \in H^2_d, \sum_{n} ||f_n||^2 < \infty \right\}$$

Thm (Greene-R-Sundberg)

$$\sum_{n} |\varphi_n(z)|^2 = 1 \ a.e. \ z \in \partial \mathbb{B}_d$$
Defn

$$\overline{Z(\mathcal{M})} = Z(\mathcal{M}) \cup \{w \in \partial \mathbb{B}_d : \liminf_{z \to w} \sum_n |\varphi_n(z)|^2 = 0\}$$

Conjecture:

$$\overline{Z(\mathcal{M})} = \sigma(T)$$

Question

If $w \in \sigma(T) \cap \partial \mathbb{B}_d$, then what can one conclude about the functions in $\mathcal{M}$ near $w$?
**Defn**

$f \in H^2_d$ is called cyclic, if

$$[f] = \{ pf : p \text{ a polynomial } \} = H^2_d.$$
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\]

\( f(z) = 1 \) is cyclic since polynomials are dense in \( H^2_d \)
\[ f \text{ is cyclic } \iff \exists \text{ polys } p_n \text{ such that } p_n f \to 1. \]

Hence, of course: If \( f \) is cyclic, then \( f(z) \neq 0 \) for all \( z \in B_d \).

\( f(z_1, ..., z_d) = 1 - z_1 \) is cyclic in \( H^2_d \).
The $H^2_d$-norm

$$k_w(z) = \frac{1}{1 - \langle z, w \rangle} = \sum_{n \geq 0} \langle z, w \rangle^n = \sum_{n \geq 0} \sum_{|\alpha| = n} \frac{|\alpha|!}{\alpha!} z^\alpha \overline{w}^\alpha$$

If $f \in \text{Hol}(B_d)$, then

$$f(z) = \sum_{n \geq 0} \left( \sum_{|\alpha| = n} \hat{f}(\alpha) z^\alpha \right)$$

and

$$\|f\|^2_{H^2_d} = \sum_{n \geq 0} \sum_{|\alpha| = n} \frac{\alpha!}{|\alpha|!} |\hat{f}(\alpha)|^2$$
Slice functions

If $f \in H^2_d$, set $f_1(\lambda) = f(\lambda, 0, ..., 0)$, then

$$
\|f_1\|_{H^2_d(D)}^2 = \sum_{n \geq 0} |\hat{f}(n, 0, ..., 0)|^2
$$

$$
\leq \sum_{n \geq 0} \sum_{|\alpha| = n} \frac{\alpha!}{|\alpha|!} |\hat{f}(\alpha)|^2 = \|f\|_{H^2_d}^2
$$

Hence $f_1 \in H^2_d(D)$ and, if $f$ is cyclic in $H^2_d$, then

$$
\|p_n^{1,1} f_1 - 1\|_{H^2_d(D)} \leq \|p_n^{f} - 1\|_{H^2_d} \to 0
$$

Thus, $f_1$ is outer in $H^2_d(D)$. 
Slice functions

If \( f \in H^2_d \), set \( f_1(\lambda) = f(\lambda, 0, \ldots, 0) \), then

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\|f_1\|_{H^2(\mathbb{D})}^2 = \sum_{n \geq 0} |\hat{f}(n, 0, \ldots, 0)|^2 \\
\leq \sum_{n \geq 0} \sum_{|\alpha| = n} \frac{\alpha!}{|\alpha|!} |\hat{f}(\alpha)|^2 = \|f\|_{H^2_d}^2
\]

Hence \( f_1 \in H^2(\mathbb{D}) \) and, if \( f \) is cyclic in \( H^2_d \), then

\[
\|p_{n,1}f_1 - 1\|_{H^2(\mathbb{D})}^2 \leq \|p_{n}f - 1\|_{H^2_d}^2 \to 0
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Thus, \( f_1 \) is outer in \( H^2(\mathbb{D}) \).
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**Thm**
If $f \in H^2_d$ is cyclic, then for all $z \in \partial B_d$

$$f_z \text{ is outer in } H^2(\mathbb{D}).$$

$$f_z(\lambda) = f(\lambda z), \quad \lambda z = (\lambda z_1, ..., \lambda z_d)$$
$H^2_d$ is invariant under composition with unitary maps $U : \mathbb{C}_d \to \mathbb{C}_d$

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**Thm**

If $f \in H^2_d$ is cyclic, then $f$ must be cyclic in $P^2(\mu)$ for every representing measure for the ball algebra.

$$\int_{\partial B_d} pd\mu = p(0), \quad \mu \geq 0,$$
The converse is false.

Take functions with Taylor coefficients supported on the diagonal:

\[ f(z_1, \ldots, z_d) = \sum_{\alpha} \hat{f}(\alpha) z_1^{\alpha_1} \cdots z_d^{\alpha_d} \]

\[ \hat{f}(\alpha) = 0 \text{ unless } \alpha_1 = \alpha_2 = \cdots = \alpha_d. \]
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Let \( R_d = d^{d/2} \), then

\[ |z|^2 = \sum_{j=1}^{d} |z_j|^2 < 1 \Rightarrow R_d|z_1z_2\cdots z_d| < 1 \]
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If \( g \in \text{Hol}(\mathbb{D}) \), then set

\[ f(z) = T(g)(z_1, \ldots, z_d) = g(R_d z_1 z_2 \cdots z_d) \in \text{Hol}(\mathbb{B}_d) \]
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\[ f(z) = T(g)(z_1, \ldots, z_d) = g(R_d z_1 z_2 \cdots z_d) \in \text{Hol}(\mathbb{B}_d) \]

\[ \|f\|_{H_d^2}^2 = \|T(g)\|_{H_d^2}^2 \approx \sum_{n \geq 0} (n + 1)^{\frac{d-1}{2}} |\hat{g}(n)|^2 = \|g\|_{D_{d-1}^{\frac{1}{2}}}^2 \]
If \( p \) is any polynomial in \( d \) variables with diagonal terms \( p_{\text{diag}} = p_{\text{diag}}(z_1 \cdots z_d) \), then

\[
\|pT(g) - 1\|_{H^2_d}^2 \geq \|p_{\text{diag}}T(g) - 1\|_{H^2_d}^2 \approx \|p_{\text{diag}}g - 1\|_{D_{d-1}^2}^2
\]
If $p$ is any polynomial in $d$ variables with diagonal terms $p_{\text{diag}} = p_{\text{diag}}(z_1 \cdots z_d)$, then

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Prop

$g$ is cyclic in $D_{\frac{d-1}{2}}$ if and only if $T(g)$ is cyclic in $H_d^2$. 
If \( p \) is any polynomial in \( d \) variables with diagonal terms \( p_{\text{diag}} = p_{\text{diag}}(z_1 \cdots z_d) \), then

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\]

**Prop**

\( g \) is cyclic in \( D_{\frac{d-1}{2}} \) if and only if \( T(g) \) is cyclic in \( H_d^2 \).

**Cor**

For \( d \geq 2 \) there is a noncyclic \( f \in H_d^2 \) such that \( f_z \) is outer in \( H^2(\mathbb{D}) \) for all \( z \in \partial B_d \).
Polynomials

If \( \|g\|_{D^{d-1/2}}^2 = \sum_{n \geq 0} (n + 1)^{d-1/2} |\hat{g}(n)|^2 \), then \( D^{d-1/2} \subseteq A(\overline{\mathbb{D}}) \) if \( d \geq 4 \)

Hence

Thm

For \( d \geq 4 \) there is a noncyclic polynomial \( f \) in \( H^2_d \), which satisfies

\[
f(z) = 1 - R_d z^1 \cdots z^d \]

\( g(z) = 1 - z \) is not cyclic in \( D^{d-1/2} \), if and only if \( d \geq 4 \).

Thm

If \( d \leq 2 \), and if \( f \) is a polynomial, then \( f \) is cyclic in \( H^2_d \) if and only if \( f(z) = 0 \) for all \( z \in B^d \).

Question

For \( d = 3 \), is a polynomial \( p \) cyclic in \( H^2_3 \) if and only if \( p(z) = 0 \) on \( B^3 \)?
Polynomials

If $\|g\|_{D_{d-1}^{d-1}}^2 = \sum_{n \geq 0} (n + 1)^{d-1} |\hat{g}(n)|^2$, then $D_{\frac{d-1}{2}} \subseteq A(\mathbb{D})$ if $d \geq 4$

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Thm

For $d \geq 4$ there is a noncyclic polynomial $f$ in $H^2_d$, which satisfies $f(z) \neq 0$ for all $z \in \mathbb{B}_d$. 

Question

For $d = 3$, is a polynomial $p$ cyclic in $H^2_3$ if and only if $p$, $0$ on $\mathbb{B}_d$?
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If \( \|g\|_{D_{d-1}}^2 = \sum_{n \geq 0} (n + 1)^{d-1} |\hat{g}(n)|^2 \), then \( D_{d-1} \subseteq A(\overline{D}) \) if \( d \geq 4 \)

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**Thm**

*For \( d \geq 4 \) there is a noncyclic polynomial \( f \) in \( H_d^2 \), which satisfies \( f(z) \neq 0 \) for all \( z \in \mathbb{B}_d \).*

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f(z) = 1 - R_d z_1 \cdots z_d
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If $\|g\|_{D^{d-1}_{\frac{d-1}{2}}}^2 = \sum_{n \geq 0}(n+1)^{\frac{d-1}{2}}|\hat{g}(n)|^2$, then $D^{d-1}_{\frac{d-1}{2}} \subseteq A(\overline{D})$ if $d \geq 4$

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For $d \geq 4$ there is a noncyclic polynomial $f$ in $H^2_d$, which satisfies $f(z) \neq 0$ for all $z \in \mathbb{B}_d$.

$$f(z) = 1 - R_d z_1 \cdots z_d$$

$g(z) = 1 - z$ is not cyclic in $D^{d-1}_{\frac{d-1}{2}}$, if and only if $d \geq 4$. 

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If \( \|g\|_{D_{\frac{d-1}{2}}}^2 = \sum_{n \geq 0} (n + 1)^{\frac{d-1}{2}} |\hat{g}(n)|^2 \), then \( D_{\frac{d-1}{2}} \subseteq A(\overline{D}) \) if \( d \geq 4 \)

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Question
For \( d = 3 \), is a polynomial \( p \) cyclic in \( H^2_3 \) if and only if \( p \neq 0 \) on \( B_d \)?
Thm (Cascante-Ortega, 1995)

Let \( 1 < \tau \leq d \)

\[
D_{\tau}(\zeta) = \{z \in \mathbb{B}_d : |1 - \langle z, \zeta \rangle|^\tau < 1 - |z|\}.
\]

Then \( M_{\tau}f \in L^2(\mu) \) for all \( f \in H^2_d \) and all \( \mu \) such that

\[
\mu(B(\zeta, \delta)) = O(\delta^\tau)
\]

where

\[
B(\zeta, \delta) = \{w \in \partial \mathbb{B}_d : |1 - \langle w, \zeta \rangle| < \delta\} \text{ (non-isotropic ball)}
\]
Thm (Cascante-Ortega, 1995)

Let $1 < \tau \leq d$

$$D_\tau(\zeta) = \{z \in \mathbb{B}_d : |1 - \langle z, \zeta \rangle|^\tau < 1 - |z|\}.$$  

Then $M_\tau f \in L^2(\mu)$ for all $f \in H^2_d$ and all $\mu$ such that

$$\mu(B(\zeta, \delta)) = O(\delta^\tau)$$

where

$$B(\zeta, \delta) = \{w \in \partial \mathbb{B}_d : |1 - \langle w, \zeta \rangle| < \delta\} \text{ (non-isotropic ball)}$$

Cor

If

$$Z(f) = \{\zeta \in \partial \mathbb{B}_d : D_\tau \lim_{z \to \zeta} f(z) = 0\}$$

and if $\mu(Z(f)) > 0$ for any $\mu$ as above, then $f$ is not cyclic in $H^2_d$. 
Cyclicity should depend on growth and smallness of $f$ near $\partial B_d$. 

Question: If $f \in H^2_d$ and if $|f(z)| \geq c > 0$ for all $z \in B_d$, then must $f$ be cyclic? Known to be true, if $f \in M(H^2_d)$, then $1/f \in M(H^2_d)$ by Corona Thm, hence $f \rightarrow 1$ - Costea, Sawyer, Wick (full Corona) - see Jingbo Xia's website for a simple argument of the one function Corona Theorem $\quad d \leq 3$ (R-Sundberg)
Cyclicity should depend on growth and smallness of $f$ near $\partial B_d$.

**Question**

If $f \in H^2_d$ and if

$$|f(z)| \geq c > 0 \quad \forall \quad z \in B_d,$$

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Cyclicity should depend on growth and smallness of \( f \) near \( \partial B_d \).

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|f(z)| \geq c > 0 \ \forall \ z \in B_d,
\]

then must \( f \) be cyclic?

Known to be true, if

- \( f \in M(H^2_d) \), then \( 1/f \in M(H^2_d) \) by Corona Thm, hence \( \frac{f}{f_r} \to 1 \)

- Costea, Sawyer, Wick (full Corona)

- see Jingbo Xia’s website for a simple argument of the one function Corona Theorem
Cyclicity should depend on growth and smallness of $f$ near $\partial B_d$.

**Question**

If $f \in H^2_d$ and if

$$|f(z)| \geq c > 0 \ \forall \ z \in B_d,$$

then must $f$ be cyclic?

Known to be true, if

- $f \in M(H^2_d)$, then $1/f \in M(H^2_d)$ by Corona Thm, hence $\frac{f}{f_r} \to 1$
- Costea, Sawyer, Wick (full Corona)
- see Jingbo Xia’s website for a simple argument of the one function Corona Theorem
- $d \leq 3$ (R-Sundberg)
\( H^2_d \)-norm as a Besov space norm

If \( R = \sum_{i=1}^{d} z_i \frac{\partial}{\partial z_i} \), then

\[
Rz^\alpha = |\alpha| z^\alpha.
\]
$H^2_d$-norm as a Besov space norm

If $R = \sum_{i=1}^{d} z_i \frac{\partial}{\partial z_i}$, then

$$Rz^\alpha = |\alpha|z^\alpha.$$ 

Fact:

- If $d$ is even, then

$$\|f\|_{H^2_d}^2 \approx \int_{|z| < 0.5} |f|^2 dV + \|R^{d/2}f\|_{L^2_\alpha(\partial B_d)}^2$$

- If $d$ is odd, then

$$\|f\|_{H^2_d}^2 \approx \int_{|z| < 0.5} |f|^2 dV + \|R^{(d-1)/2}f\|_{H^2(\partial B_d)}^2$$
$H^2_d$-norm as a Besov space norm

If $R = \sum_{i=1}^{d} z_i \partial \frac{\partial}{\partial z_i}$, then

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Fact:

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Thus, if $d = 2, 3$, then we only need to consider $Rf$. 
Thm ($d \leq 3$)

If $f, g \in H^2_d$, if $|g(z)| \leq |f(z)| \ \forall z$, then $g \in [f]$
Thm ($d \leq 3$)
If $f, g \in H^2_d$,
\[ \text{if } |g(z)| \leq |f(z)| \forall z, \text{ then } g \in [f] \]

Strategy:
- $\varphi = g/f \in H^\infty$
- $\varphi_r f \in [f]$ for all $r < 1$
- $\varphi_r(z)f(z) \to g(z)$ in $\mathbb{B}_d$
- Need $\|\varphi_r f\|_{H^2_d} \leq C$
The proof of $\| \varphi_r f \|_{H_d^2} \leq c$ for $d = 2, 3$

$\| \| \| \|$ norms are $H^2(\partial B_d)$- or $L^2_a(B_d)$-norms.

Facts:
- $\|f - f_r\| \leq (1 - r) \|Rf\|$
- $\|R \varphi_r\|_\infty \leq \frac{\|\varphi\|_\infty}{1 - r}$
The proof of $\|\varphi_{rf}\|_{H^2_d} \leq c$ for $d = 2, 3$

$\| \|$ norms are $H^2(\partial B_d)$- or $L^2_a(B_d)$-norms.

Facts:

- $\|f - f_r\| \leq (1 - r) \|Rf\|
- $\|R \varphi_r\|_\infty \leq \frac{\|\varphi\|_\infty}{1 - r}$

\[
\|R(\varphi_{rf})\| \leq \|R(\varphi_r(f - f_r))\| + \|R(\varphi_{rf_r})\|, \quad \varphi f = g \\
\leq \|(R \varphi_r)(f - f_r)\| + \|\varphi_r R(f - f_r)\| + \|Rg\| \\
\leq \frac{\|\varphi\|_\infty}{1 - r} \|f - f_r\| + \|\varphi\|_\infty \|R(f - f_r)\| + \|g\| \\
\leq \frac{\|\varphi\|_\infty}{1 - r} (1 - r) \|Rf\| + M \leq M'
\]
J. Xia’s argument:

Suppose \( f \in M(H_d^2) \), \( 1/f \in H^\infty(\mathbb{B}_d) \)

to show: \( 1/f \in M(H_d^2) \)

\( d = 2, 3 \)

\( R \left( \frac{g}{f} \right) = \frac{fRg - gRf}{f^2} \)

\( R(fg) = gRf + fRg \)

\( R \left( \frac{g}{f} \right) = 2 \frac{Rg}{f} - \frac{R(fg)}{f^2} \)
J. Xia’s argument:

Suppose $f \in M(H^2_d)$, $1/f \in H^\infty(\mathbb{B}_d)$

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- $R(fg) = gRf + fRg$
- $R \left( \frac{g}{f} \right) = 2 \frac{Rg}{f} - \frac{R(fg)}{f^2}$

$d = 4, 5$

$$R^2 \left( \frac{g}{f} \right) = 4 \frac{R^2g}{f} - 2 \frac{R(fRg)}{f^2} - 3 \frac{R^2(fg)}{f^2} + 2 \frac{R(fR(fg))}{f^3}$$