PLANAR ZERO-DIVISOR GRAPHS

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ABSTRACT. Given any commutative ring $R$, one can associate with $R$ an undirected graph $\Gamma(R)$ whose vertices are the nonzero zero-divisors of $R$, and two distinct vertices $x$ and $y$ are joined by an edge if $xy = 0$. Anderson, Frazier, Lauve, and Livingston asked which finite commutative rings $R$ have planar zero-divisor graphs. In this paper, we classify all such rings.

1. Introduction

The notion of the zero-divisor graph $\Gamma(R)$ of a commutative ring $R$ was introduced by Beck in [6] and has been studied in [1], [2], [3], [4], [5], [8], [9], and [10]; it has been extended to commutative semigroups in [7] and noncommutative rings in [11]. The vertices of $\Gamma(R)$ are the nonzero zero-divisors of $R$, and two distinct vertices $x$ and $y$ are joined by an edge if and only if $xy = 0$. Thus $\Gamma(R) = \emptyset$ if and only if $R$ is an integral domain. It is known that $\Gamma(R)$ is connected, with diameter $\leq 3$ and girth $\leq 4$ (provided $\Gamma(R)$ has a cycle), and that $\Gamma(R)$ is finite and nonempty if and only if $R$ is finite and not a field (see [3] and [4]). In [3], Anderson, Frazier, Lauve, and Livingston determined which rings of the form $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ have planar zero-divisor graphs and posed the question as to which finite commutative rings have planar zero-divisor graphs. In this paper, some of which will appear in the author’s Ph.D. thesis, we will classify all such rings (Theorem 3.7). We also obtain several results which are interesting in their own right; for instance, a (finite) commutative local ring (not a field) with more than 27 elements (or more than 9 zero-divisors) cannot have a planar zero-divisor graph.

In this paper, $R$ will always denote a commutative ring (with $1 \neq 0$) and $\Gamma(R)$ will denote its zero-divisor graph. $Z(R)$ will denote the set of zero-divisors of $R$. We will often speak of cliques, complete subgraphs of $\Gamma(R)$. The clique number of a graph $G$ is the size of the largest vertex set that induces a clique in $G$. We will typically denote the clique number of $\Gamma(R)$ by $\omega(R)$. Observe that the clique number of an integral domain is zero. As usual, $\mathbb{Z}$, $\mathbb{Z}_n$, and $\mathbb{F}_m$ will denote the integers, the integers mod $n$, and the field with $m$ elements, respectively. Also, $k$ or $k_5$ will denote a finite field. $K_n$ and $K_{m,n}$ will denote complete graphs on $n$ vertices and complete bipartite graphs with a bipartition into vertex sets of cardinality $m$ and $n$, respectively.

It is also worth mentioning that in [6] and [2], a slightly different definition of the zero-divisor graph is used, as each element of $R$ is allowed to be a vertex in the graph. However, one can easily show that the clique number of said graph is $\omega(R) + 1$ (with $\omega(R)$ as defined above)(cf. [3, Section 3]).

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In [1], Akbari, Maimani, and Yassemi have also investigated when $\Gamma(R)$ is planar, and some of the results in our paper parallel some of theirs. However, our interest lies mainly in classifying those rings whose zero-divisor graphs are planar, which is not explicitly done in their paper.

Since $R$ is finite, we may write $R$ as the product of finitely many local rings. So, we shall consider two cases, depending on whether $R$ is local or $R$ decomposes into a product of two or more finite local rings. These cases are covered separately in sections two and three, respectively.

2. The non-local case

In this case, with our approach we will also be able to classify those (finite, non-local) rings $R$ with $\omega(R) = 4$ as a bonus. To this end, we will want to note which finite commutative rings $R$ have $\omega(R) \leq 3$. We will also have cause to make use of the famous theorem of Kuratowski. Note that $\omega(R) = 0$ if and only if $R$ is an integral domain.

**Lemma 2.1.** (Beck) Let $R$ be a finite commutative ring. Then $\omega(R) = 1$ if and only if $R$ is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$.

**Proof.** A proof can be found in [6, Proposition 2.2].

**Lemma 2.2.** (Beck) Let $R$ be a finite commutative ring, and $k_1$ and $k_2$ finite fields.

Then $\omega(R) = 2$ if and only if $R$ is isomorphic to one of:

- $\mathbb{Z}_9, \mathbb{Z}_3[X]/(X^2), \mathbb{Z}_8, \mathbb{Z}_2[X]/(X^3), \mathbb{Z}_4[X]/(2X, X^2 - 2), k_1 \times \mathbb{Z}_4, k_1 \times \mathbb{Z}_2[X]/(X^2)$, or $k_1 \times k_2$.

**Proof.** A proof can be found in [6, page 226].

**Lemma 2.3.** (D.D. Anderson and Nasseer) Let $R$ be a finite commutative ring, and $k_1, k_2$, and $k_3$ finite fields. Then $\omega(R) = 3$ if and only if $R$ is isomorphic to one of:

- $\mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_4 \times \mathbb{Z}_2[X]/(X^2), \mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2[X]/(X^2), k_1 \times k_2 \times k_3$,
- $k_1 \times k_2 \times \mathbb{Z}_4, k_1 \times k_2 \times \mathbb{Z}_2[X]/(X^2), k_1 \times \mathbb{Z}_8$,
- $k_1 \times \mathbb{Z}_9, k_1 \times \mathbb{Z}_3[X]/(X^2), k_1 \times \mathbb{Z}_2[X]/(X^3)$,
- $k_1 \times \mathbb{Z}_4[X]/(2X, X^2 - 2), \mathbb{Z}_4[X]/(X^4)$,
- $\mathbb{Z}_4[X]/(X^2 + X + 1), \mathbb{Z}_2[X, Y]/(X, Y^2, \mathbb{Z}_4[X]/(2, X^2), \mathbb{Z}_2[X]/(X^3), \mathbb{Z}_9[X]/(X^2 - 3, 3X), \mathbb{Z}_6[X]/(X^2 - 6, 3X), \mathbb{Z}_2[X, Y]/(X^2, Y^2 - XY), \mathbb{Z}_2[X, Y]/(X^2, Y^2), \mathbb{Z}_6[X]/(2X - 4, X^2), \mathbb{Z}_4[X]/(X^2), \mathbb{Z}_4[X]/(X^2 - 2X)$,
- $\mathbb{Z}_4[X]/(X^2, XY - 2, Y^2 - XY, 2X, 2Y), or \mathbb{Z}_4[X, Y]/(X^2, XY - 2, Y^2, 2X, 2Y)$.

**Proof.** A proof can be found in [2, Theorem 4.4].

As mentioned previously, the above three results are stated somewhat differently in the cited papers since those authors used a slightly different definition of $\Gamma(R)$.

However, it is easy to verify that they translate as above (cf. [3, Example 3.1 and Theorem 3.2]). For other results on $\omega(R)$, see [6], [2], and [3, Section 3].

We will often use the following famous result as well; a proof can be found in [12, pp. 259–264]. One immediate consequence of Theorem 2.4 will be that if $\omega(R) \geq 5$, then $\Gamma(R)$ is not planar.
Theorem 2.4. (Kuratowski) A graph is planar if and only if it contains no subdivision homeomorphic to $K_5$ or $K_{3,3}$.

Since we will be considering the zero-divisor graph of a finite product, we need to know how the clique number behaves with respect to products. This is addressed in the following result.

Proposition 2.5. Let $R_1, \ldots, R_n$ be finite commutative rings, $R$ a finite commutative ring which is not an integral domain, and $D$ an integral domain. Then:

(a) $\omega(R_1 \times \cdots \times R_n) \geq \omega(R_1) + \cdots + \omega(R_n)$.

(b) $\omega(R \times D) = \omega(R) + 1$.

Proof. To prove (a), we may take $n = 2$, the general result follows by induction. Say $\omega(R_1) = j$ and $\omega(R_2) = l$, and let $\{r_1, \ldots, r_j\}$ be a $j$-clique in $R_1$ and $\{s_1, \ldots, s_l\}$ be an $l$-clique in $R_2$. Then

$$\{(r_i, 0) \mid i = 1, \ldots, j\} \cup \{(0, s_i) \mid i = 1, \ldots, l\}$$

is a clique in $R_1 \times R_2$; so $\omega(R_1 \times R_2) \geq \omega(R_1) + \omega(R_2)$.

To see (b), say $\omega(R) = m$, and let $\{x_1, \ldots, x_m\}$ be an $m$-clique in $R$. Then

$$\{(x_i, 0) \mid i = 1, \ldots, m\} \cup \{(0, 1)\}$$

is an $m + 1$ clique in $R \times D$; so $\omega(R \times D) \geq m + 1$. To see that strict inequality is impossible, note that since $D$ is a domain, a clique can not contain two elements of the form $(x, d_1), (y, d_2)$, where $d_1$ and $d_2$ are two nonzero elements of $D$. Thus an $m + 2$ clique in $R \times D$ would give rise to an $m + 1$ clique in $R$, contrary to $\omega(R) = m$. 

Note that strict inequality may hold in (a) above. For example, $\omega(Z_4) = 1$, but $\omega(Z_4 \times Z_4) = 3$. This example motivates a slight strengthening of the above proposition.

Proposition 2.6. Let $R$ and $S$ be finite commutative rings, $\omega(R) = m$ with $\{r_1, \ldots, r_m\}$ an $m$-clique in $R$, and $\omega(S) = n$ with $\{s_1, \ldots, s_n\}$ an $n$-clique in $S$. If $r_i^2 = 0$ for some $i$ and $s_j^2 = 0$ for some $j$, then $\omega(R \times S) > \omega(R) + \omega(S)$.

Proof. Certainly “$\geq$” holds since

$$C = \{(r_1, 0), \ldots, (r_m, 0), (0, s_1), \ldots, (0, s_n)\}$$

is an $m + n$ clique. However, since $r_ir_k = 0$ for all $k$, and $s_js_k = 0$ for all $k$, we see that $C \cup \{(r_i, s_j)\}$ is an $m + n + 1$ clique in $R \times S$. 

It is also possible to strengthen the above result somewhat further by considering the number of 2-nilpotents, but we will not need this in our paper. To finish the preliminaries, we state an often-used corollary.

Corollary 2.7. Let $n \geq 2$ be an integer and $R_1, \ldots, R_n$ be finite commutative rings. Then $\omega(R_1 \times \cdots \times R_n) \geq n$ with equality if and only if either

(a) each $R_i$ is a finite field, or

(b) exactly one factor is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, and all the other factors are finite fields.

Proof. The key observation is that if $R$ and $S$ are integral domains, then $\Gamma(R \times S)$ is a complete bipartite graph, and thus $\omega(R \times S) = 2$. The corollary then follows from this observation and Propositions 2.5 and 2.6.
For the remainder of this section, \( R \) will denote a finite, non-local ring, and \( R_1 \times \cdots \times R_n \) will be its local decomposition.

We will prove the main result of this section (Theorem 2.12) via a handful of somewhat technical lemmas.

**Lemma 2.8.** Let \( R = R_1 \times R_2 \times R_3 \), where each \( R_i \) is a finite, commutative, local ring, and let \( k_1 \) and \( k_2 \) be finite fields. If \( \omega(R) = 4 \), then \( R \) is isomorphic to either

(a) \( k_1 \times (\text{one of the rings in Lemma 2.1}) \times (\text{one of the rings in Lemma 2.1}) \), or

(b) \( k_1 \times k_2 \times (\text{one of the local rings in Lemma 2.2}) \).

**Proof.** Since \( \omega(R) = 4 \), not all the factors can be fields. If exactly one factor, say \( R_1 \), is a field, then

\[
\omega(R_1 \times R_2 \times R_3) = \omega(R_2 \times R_3) + 1.
\]

Since any local ring with clique number 2 contains a maximal clique with a 2-nilpotent (see Lemma 2.2), we must have \( \omega(R_2) = \omega(R_3) = 1 \). Thus \( R_2 \) and \( R_3 \) must be isomorphic to either \( \mathbb{Z}_4 \) or \( \mathbb{Z}_2[X]/(X^2) \). Then, one can see that

\[
\omega(k_1 \times \mathbb{Z}_4[X]/(X^2) \times \mathbb{Z}_2[X]/(X^2)) = \omega(k_1 \times \mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_4) = \omega(k_1 \times \mathbb{Z}_2 \times \mathbb{Z}_4) = 4.
\]

If exactly two factors are fields, similar reasoning shows that \( R \) is isomorphic to \( k_1 \times k_2 \times (\text{one of the local rings listed in Lemma 2.2}) \). If none of the factors is a field, then since each of the local rings listed in Lemmas 2.1-2.3 contains a maximal clique with 2-nilpotents, we must have each \( R_i \) isomorphic to \( \mathbb{Z}_4 \) or \( \mathbb{Z}_2[X]/(X^2) \), since each of the \( R_i \)'s must have clique number 1. But then \( \omega(R_1 \times R_2 \times R_3) > 4 \).

Thus the only possibilities for \( R \) are as stated. \( \square \)

**Lemma 2.9.** Let \( R = R_1 \times R_2 \), where each \( R_i \) is a finite, commutative, local ring, and let \( k \) be a finite field. If \( \omega(R) = 4 \), then \( R \) is isomorphic to either

(a) \( k \times (\text{one of the local rings in Lemma 2.3}) \), or

(b) \( (\text{one of the rings in Lemma 2.1}) \times (\mathbb{Z}_2, \mathbb{Z}_2[X]/(X^3)) \), or \( \mathbb{Z}_4[X]/(2X, X^2 - 2) \).

**Proof.** Certainly, any ring \( R \) as in (a) has clique number 4.

By Propositions 2.5 and 2.6, we are left to consider only those \( R_1 \) and \( R_2 \) (not fields) with \( \omega(R_1) + \omega(R_2) = 3 \), that is, only those \( R_1 \) as in Lemma 2.1, and only those local \( R_2 \) as in Lemma 2.2.

We claim that \( \omega(\mathbb{Z}_4 \times \mathbb{Z}_8) = 4 \). To see this, the subgraph induced by

\[
((\mathbb{Z}_4 - \{0\}) \times \{0\}) \cup \{0\} \times (\mathbb{Z}_8 - \{0\})
\]

contains a \( K_{3,7} \). Any zero-divisor of the form \((2, x)\), where \( x \in (\mathbb{Z}_8 - \{0\}) \) with \( x \) odd, is adjacent only to \((2, 0)\). Similarly, \((1, 4)\) and \((3, 4)\) are adjacent only to \((0, 2), (0, 4), (0, 6)\). And, \((1, 2), (3, 2), (1, 6), (3, 6)\) are adjacent only to \((0, 4)\). Since the only other zero-divisors in \( \mathbb{Z}_4 \times \mathbb{Z}_8 \) capable of contributing to a clique with more than 2 elements are \((2, 2), (2, 4), (2, 6), \) then certainly

\[
\omega(\mathbb{Z}_4 \times \mathbb{Z}_8) \leq 5.
\]

However, since in \( \mathbb{Z}_8 \), \( ann(2) = ann(6) = \{0, 4\} \), we see that

\[
\omega(\mathbb{Z}_4 \times \mathbb{Z}_8) \leq 4.
\]

Since \((2, 0), (2, 2), (0, 4), (2, 4)\) is a 4-clique, we conclude that

\[
\omega(\mathbb{Z}_4 \times \mathbb{Z}_8) = 4.
\]

Similarly, \( \omega(\mathbb{Z}_4 \times (\mathbb{Z}_2[X]/(X^3) \text{ or } \mathbb{Z}_4[X]/(2X, X^2 - 2))) = 4 \) and

\[
\omega(\mathbb{Z}_4[X]/(X^2) \times (\mathbb{Z}_8, \mathbb{Z}_2[X]/(X^3)) \text{ or } \mathbb{Z}_4[X]/(2X, X^2 - 2))) = 4.
\]

However, \( \omega(\mathbb{Z}_4 \times \mathbb{Z}_8) \geq 5 \), as \( \{(2, 0), (0, 3), (0, 6), (2, 3), (2, 6)\} \) is a 5-clique. Similarly, \( \mathbb{Z}_4 \times \mathbb{Z}_2[X]/(X^3) \text{ or } \mathbb{Z}_4[X]/(X^2) \times \mathbb{Z}_2[X]/(X^2) \text{ or } \mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2 \) all have clique number at least 5. \( \square \)

We next summarize our previous results.
Proposition 2.10. Let $R$ be a finite, commutative, non-local ring with $\omega(R) = 4$, and let $k_1, k_2, k_3$, and $k_4$ be finite fields. Then $R$ is isomorphic to one of the following:

(a) $k_1 \times k_2 \times k_3 \times k_4$,
(b) $k_1 \times (\text{one of the rings in Lemma 2.1}) \times (\text{one of the rings in Lemma 2.1})$,
(c) $k_1 \times k_2 \times (\text{one of the local rings in Lemma 2.2})$,
(d) $k_1 \times (\text{one of the local rings in Lemma 2.3})$,
(e) $(\text{one of the rings in Lemma 2.1}) \times (\mathbb{Z}_p, \mathbb{Z}_2[X]/(X^3))$, or $\mathbb{Z}_4[X]/(2X, X^2 - 2)$, or

or

(f) $k_1 \times k_2 \times k_3 \times (\text{one of the rings in Lemma 2.1})$.

Some remarks are in order. Virtually all of the rings listed above can immediately be seen to have nonplanar zero-divisor graphs. Observe that if $R = R_1 \times R_2$, then $\Gamma(R)$ contains a $K_{|R_1|-1, |R_2|-1}$. Hence, if $R_1$ and $R_2$ each have 4 or more elements, then $\Gamma(R_1 \times R_2)$ contains a $K_{3,3}$, and will thus (by Kuratowski’s Theorem) be nonplanar. Hence the only rings listed above whose zero-divisor graphs could possibly be planar are found in part (d) of the above proposition. However, we will show that the zero-divisor graphs of these rings all fail to be planar as well. To this end, we will need the following easy, but useful, lemma.

Lemma 2.11. Let $R$ be a commutative ring and $D$ an integral domain. If $\Gamma(R)$ contains a 3-clique $C$ with the property that for all $x, y \in C$ (and $x$ and $y$ not necessarily distinct) we have $xy = 0$, then $\Gamma(D \times R)$ is not planar.

Proof. If $\{x, y, z\}$ is such a 3-clique, then the subgraph of $\Gamma(D \times R)$ spanned by $\{(0, x), (0, y), (0, z), (1, x), (1, y), (1, z)\}$ contains a $K_{3,3}$. Thus, $\Gamma(D \times R)$ is not planar.

Since each of the local rings listed in Lemma 2.3 (with the exception of $\mathbb{Z}_{27}$, $\mathbb{Z}_9[X]/(X^3)$, $\mathbb{Z}_9[X]/(X^2 - 6, 3X)$, and $\mathbb{Z}_9[X]/(X^2 - 3, 3X)$) has such a clique, we need only consider products of $\mathbb{Z}_3$ or $\mathbb{Z}_9$ with these rings.

Fortunately, since $\mathbb{Z}_{27}, \mathbb{Z}_9[X]/(X^3), \mathbb{Z}_9[X]/(X^2 - 6, 3X)$, and $\mathbb{Z}_9[X]/(X^2 - 3, 3X)$ all have isomorphic zero-divisor graphs with corresponding 2-nilpotents, one can easily check that $k \times \mathbb{Z}_{27}$, $k \times \mathbb{Z}_9[X]/(X^3), k \times \mathbb{Z}_9[X]/(X^2 - 6, 3X)$, and $k \times \mathbb{Z}_9[X]/(X^2 - 3, 3X)$ all have isomorphic zero-divisor graphs as well.

Thus, it will suffice to examine the planarity of $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_{27})$ and $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_{27})$. However, these graphs each contain a $K_{3,3}$, and hence are nonplanar (cf. [3, Theorem 5.1]). Thus all of the rings listed in Proposition 2.10 have nonplanar zero-divisor graphs.

We are now left only to consider the planarity of the zero-divisor graphs of those rings listed in Lemmas 2.1-2.3. One can easily check that all of the zero-divisor graphs of the local rings in question have planar embeddings. By the remarks preceding Lemma 2.11, all that remains is to consider the planarity of the zero-divisor graphs of $k \times \mathbb{Z}_4$, $k \times \mathbb{Z}_2[X]/(X^2)$, $k_1 \times k_2$, $k_1 \times k_2 \times k_3$, $k \times \mathbb{Z}_9$, $k \times \mathbb{Z}_9[X]/(X^2)$, $k \times \mathbb{Z}_9[X]/(X^3)$, and $k \times \mathbb{Z}_4[X]/(2X, X^2 - 2)$, where $k$ is $\mathbb{Z}_2$ or $\mathbb{Z}_3$, and $k_1, k_2$, and $k_3$ are chosen so the zero-divisor graph does not contain a $K_{3,3}$.

It is easy to see that $\Gamma((\mathbb{Z}_2 \times \mathbb{Z}_3) \times \text{(any finite field)})$ is planar. As indicated in previous remarks, it therefore suffices to consider only the zero-divisor graphs of $k \times \mathbb{Z}_4$, $k_1 \times k_2 \times k_3$, and $k \times \mathbb{Z}_9$, where $k, k_1, k_2$, and $k_3$ are $\mathbb{Z}_2$ or $\mathbb{Z}_3$. By producing the zero-divisor graphs of these rings, one finds that only $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$,
$Z_3 \times Z_3 \times Z_3$, and $Z_3 \times Z_6$ have nonplanar zero-divisor graphs (cf. [3, Theorem 5.1]), giving us the following theorem.

**Theorem 2.12.** Let $R$ be a finite, non-local, commutative ring, and $k$ a finite field. Then $\Gamma(R)$ is planar if and only if $R$ is isomorphic to one of the following rings:

- $k \times Z_2$, $k \times Z_3$,
- $Z_2 \times Z_4$, $Z_2 \times Z_2[X]/(X^2)$,
- $Z_3 \times Z_4$, $Z_3 \times Z_2[X]/(X^2)$,
- $Z_2 \times Z_2 \times Z_2$, $Z_2 \times Z_2 \times Z_3$,
- $Z_2 \times Z_6$, $Z_2 \times Z_3[X]/(X^2)$,
- $Z_3 \times Z_9$, $Z_3 \times Z_3[X]/(X^2)$,
- $Z_2 \times Z_8$, $Z_2 \times Z_2[X]/(X^2)$, or $Z_2 \times Z_4[X]/(2X, X^2 - 2)$.

It is interesting to note that all of the rings listed in Theorem 2.12 have cardinality either $p^n$ or $p^n + 1$ for some prime integer $p$ and integer $n \geq 1$. Conversely, note that $|\Gamma(F_{p^n} \times Z_2)| = p^n$ and $|\Gamma(F_{p^n} \times Z_3)| = p^n + 1$.

3. The Local Case

We begin by fixing some notation. In this section, unless otherwise indicated, $(R, M)$ will denote a (commutative) finite local ring with (unique) nonzero maximal ideal $M$. And, since $R$ is finite, there is an integer $n \geq 2$ so that $M^n = 0$ and $M^{n-1} \neq 0$. Thus $M^i/M^{i+1}$ is an $R/M$-vector space for each $1 \leq i \leq n - 1$, and hence $|M^{n-1}| = |R/M|^d$, where $d = \dim_{R/M} M^{n-1}$. Our first result gives a bound on $n$.

**Proposition 3.1.** Let $(R, M)$ be a finite, commutative local ring. If $\Gamma(R)$ is planar, then $n$ (as above) satisfies $n \leq 5$.

**Proof.** Note that if $n \geq 6$, then $2n - 6 \geq n$; so that $(M^{n-3})^2 = 0$, i.e., $M^{n-3} - \{0\}$ is a clique. Therefore, if $\Gamma(R)$ is to be planar, we must have $|M^{n-3}| \leq 5$, else $\Gamma(R)$ will contain a $K_5$, and will thus be nonplanar. But

$$|M^{n-3}| = |M^{n-3}/M^{n-2}||M^{n-2}/M^{n-1}||M^{n-1}/M^n| \geq 2^3 = 8,$$

which is a contradiction. \qed

As in the previous section, we will obtain the main result via a sequence of cases, as to whether $n$ is 2, 3, 4, or 5.

**Lemma 3.2.** If $(R, M)$ is a finite, commutative local ring with $0 = M^5 \subsetneq M^4 \subsetneq \cdots \subsetneq M$, then $\Gamma(R)$ is nonplanar.

**Proof.** Suppose that $\Gamma(R)$ is planar. Then $(M^3)^2 = 0$, so $M^3 - \{0\}$ is a clique. Thus $|M^3| = 2, 3, 4, \text{ or } 5$. Since $M^4 \neq 0$, necessarily $|M^4| \geq 2$. Since $|M^4| \mid |M^3|$, necessarily $|M^3| = 4$. But $M^2M^3 = 0$. Since $|M^3 - \{0\}| = 3$, and $|M^2 - M^3| \geq 3$ in any event, $\Gamma(R)$ will thus contain a $K_{3,3}$, and hence will be nonplanar by Kuratowski’s Theorem, a contradiction. \qed

The $n = 3$ and $n = 4$ cases are somewhat involved, but the $n = 2$ case is easy with the help of the following proposition from [2, Proposition 4.2].

**Proposition 3.3.** Let $(R, M)$ be a finite, commutative local ring, not a field, with $M^2 = 0$, char$(R/M) = p$, and $s = \dim_{R/M} M$.

(a) If char$(R) = p$, then $R \cong (R/M)[X_1, \ldots, X_s]/(X_1^2, \ldots, X_s^2)$.
(b) If \( \text{char}(R) = p^2 \), then \( R \cong \mathbb{Z}_p[X_1, \ldots, X_3]/((p, X_2, \ldots, X_3)^2 + (f(X_1))) \), where \( f(X_1) \in \mathbb{Z}_p[X_1] \) is irreducible mod \( p \) with \( \mathbb{Z}_p[X_1]/(f(X_1)) \cong R/M \).

As an immediate corollary of the above, we have:

**Corollary 3.4.** Let \((R, M)\) be a finite, commutative local ring with \( \omega(R) = 4 \) and \( M^2 = 0 \). Then \( R \) is isomorphic to either \( \mathbb{Z}_{25} \) or \( \mathbb{Z}_5[X]/(X^2) \). Thus \( \Gamma(R) = K_4 \), and hence \( \Gamma(R) \) is planar.

**Proof.** First, observe that \( \Gamma(R) \) is complete since \( M^2 = 0 \) (and since the vertices of \( \Gamma(R) \) are precisely the nonzero elements of \( M \)). Further, \( \omega(R) = 4 = |M - \{0\}| \). Thus we have \( 5 = |M| = |R/M|^4 \), which in turn yields \( s = 1 \) and \( |R/M| = 5 \). It then follows that \( |R| = 25 \), and from the previous proposition, the only possibilities for \( R \) can be the two given rings, as \( \text{char}(R) = 5 \) or \( \text{char}(R) = 25 \).

Note that Proposition 3.3 can also be used to classify those finite, commutative local rings \((R, M)\) with \( M^2 = 0 \) and \( \omega(R) = 1, 2, \) or \( 3 \). Namely, \( \mathbb{Z}_4, \mathbb{Z}_2[X]/(X^2) \), \( \mathbb{Z}_8, \mathbb{Z}_4[X]/(X^2) \), \( F_4[X]/(X^2), \mathbb{Z}_4[X]/(X^2 + X + 1), \mathbb{Z}_2[X, Y]/(X, Y)^2 \), and \( \mathbb{Z}_4[X]/(X, Y)^2 \) can be seen to have this property. Each of these rings has a planar zero-divisor graph; thus there are 16 local rings with \( M^2 = 0 \) and \( M \neq 0 \) whose zero-divisor graph is planar.

**Lemma 3.5.** Let \((R, M)\) be a finite, commutative local ring with \( 0 = M^4 \subseteq M^3 \subseteq M^2 \subsetneq M \). If \( \Gamma(R) \) is planar, then \( \omega(R) = 3 \). Thus \( R \) is isomorphic to one of the following five rings:

\[
\mathbb{Z}_{16}, \mathbb{Z}_2[X]/(X^4), \mathbb{Z}_4[X]/(2X, X^3 - 2), \mathbb{Z}_4[X]/(X^2 - 2), \text{ or } \mathbb{Z}_4[X]/(X^2 + 2X + 2).
\]

**Proof.** Since \( 0 = M^4 \subseteq M^3 \subseteq M^2 \subsetneq M \), we may pick \( x \in M - (0 : M) \). This is possible since \( M^2 \neq 0 \). We claim that

\[
((0 : M) \cup \{x\}) - \{0\}
\]

is a clique. To see this, if \( z \in (0 : M) \), then certainly \( xz = 0 \) (since \( x \in M \)), and if \( y \in (0 : M) - \{x\} \), then \( y \) is a zero-divisor; so \( yz = 0 \), and hence \( yz = 0 \). Thus \( |(0 : M)| = |((0 : M) \cup \{x\}) - \{0\}| \leq 4 \). Observe that the equality holds since \( x \notin (0 : M) \).

Now, observe that \( R/M \) injects into \((0 : M)\). To see this, pick \( 0 \neq z \in (0 : M) \), and define \( \phi : R \to (0 : M) \) by \( \phi(r) = rz \). Note that if \( r \in \ker(\phi) \), i.e., \( rz = 0 \), then \( r \) is a zero-divisor, and so \( r \in M \). Thus \( \ker(\phi) = M \), and hence

\[
R/M = R/\ker(\phi) \cong \text{im}(\phi) \subseteq (0 : M).
\]

This tells us that \( |R/M| \leq |(0 : M)| \leq 4 \). So the proof splits into cases, as \( |R/M| \) equals 2, 3, or 4.

If \( |R/M| = 3 \), then \( |M^3| \geq |R/M| = 3 \). So, certainly, \( |M^2| \geq 6 \) and \( |M^2 - \{0\}| \geq 5 \). However, \( M^2 - \{0\} \) is a clique, so that \( \Gamma(R) \) contains a \( K_5 \), and is nonplanar. The same conclusion obviously holds if \( |R/M| = 4 \).

If \( |R/M| = 2 \), then \( |M^3| = 2^d \) (for some integer \( d \geq 1 \)). If \( \Gamma(R) \) is planar, then necessarily \( |M^2| = 4 \). For 2 divides \( |M^2| \), and if \( |M^2| \geq 6 \), then as above, \( \Gamma(R) \) would contain a \( K_5 \).

Thus, necessarily, \( |M^3| = 2, |M^2| = 4 \) (and \( |M^2/M^3| = 2 \)). We claim that \( M^2 \) is principal. To see this, say

\[
M^3 = \{0, x\}, M^2 = \{0, x, b, y\}.
\]
We claim that $M^2 = Rb$. To see why, pick $m \in M$ so that $mm^2 \neq 0$. Then $mb \in M^2$, and we can take $mb = x$. Now $b + x = y$ ($b + x \neq 0$ since $b \in M^2 - M^3$, and obviously $b + x \neq x$ and $b + x \neq b$). So $y = b + x = b + mb = b(1 + m)$, and hence $M^2 = Rb$, as claimed.

Since $b \in M^2$ and $b^2 = 0$, and thus $(0 : b) \supseteq Rb = M^2$. If $Rb = (0 : b)$, then Anderson and Nasseer (see [2, p. 511]) show that $|[R]| = 16, \omega(R) = 3$, and $R$ is one of the rings in the statement of the lemma (each of which has a planar zero-divisor graph).

If $(0 : b) \supseteq Rb = M^2$, pick $t \in (0 : b) - Rb$. Then $tb = 0$, so that $t(Rb) = tM^2 = 0$. So, $M^2 \subseteq (0 : M^2) \subseteq M$, since $t \notin M^2$. Thus $|(0 : M^2)| \geq 8$ since $|M^2| = 4$. Therefore $|(0 : M^2) - M^2| \geq 4$, and one then sees that the subgraph of $\Gamma(R)$ induced by $(0 : M^2)$ and $M^2 - \{0\}$ contains a $K_{3,3}$, which forces $\Gamma(R)$ to be nonplanar.

Some of the statements in the above proof could be obtained using Nakayama’s Lemma, but we have opted to keep the arguments as elementary and self-contained as possible.

The point of our next result is somewhat similar; we will see that if $0 = M^3 \subseteq M^2 \subseteq M$ and $\Gamma(R)$ is planar, then $\omega(R) \leq 3$. In effect, if all we are concerned with is the planarity of the zero-divisor graph, we need not try to explicitly classify all the local rings with clique number 4. In fact, the 2 rings in Corollary 3.4 are the only finite local rings with $\omega(R) = 4$ and $\Gamma(R)$ planar. The next result applies when $M^3 = 0, M^2 \neq 0$, and $\omega(R) \geq 3$. One can check that precisely the local rings $\mathbb{Z}_8[X]/(X^3)$, and $\mathbb{Z}_4[X]/(2X, X^2 - 2)$ have $M^3 = 0, M^2 \neq 0$, and $\omega(R) \leq 2$, and these rings all have planar zero-divisor graphs. This fact, together with the next lemma, will give us precisely 14 local rings with $M^3 = 0$ and $M^2 \neq 0$ whose zero-divisor graphs are planar.

**Lemma 3.6.** Let $(R, M)$ be a finite, commutative local ring with $0 = M^3 \subseteq M^2 \subseteq M$ and $\omega(R) \geq 3$. If $\Gamma(R)$ is planar, then $\omega(R) = 3$ and $R$ is isomorphic to one of the following eleven rings:

- $\mathbb{Z}_{27}, \mathbb{Z}_3[X]/(X^3), \mathbb{Z}_9[X]/(X^2 - 3, 3X), \mathbb{Z}_9[X]/(X^2 - 6, 3X)\$
- $\mathbb{Z}_2[X, Y]/(X^2, Y^2 - XY), \mathbb{Z}_2[X, Y]/(X^2, 2X), \mathbb{Z}_2[X]/(2X - 4, X^2), \mathbb{Z}_4[X]/(X^2), \mathbb{Z}_4[X]/(X^2 - 2X), \mathbb{Z}_4[X, Y]/(X^2, XY - 2, Y^2 - XY, 2X, 2Y), \mathbb{Z}_4[X, Y]/(X^2, XY - 2, Y^2, 2X, 2Y)\$.

**Proof.** First, observe that each of the rings listed above has $M^3 = 0$ and $M^2 \neq 0$.

Since $M^2 - \{0\}$ is a clique, necessarily $|M^2 - \{0\}| \leq 4$, i.e., $|M^2| \leq 5$. If $|M^2| = 4$, then $|M| \geq 8$, Thus $|M - M^2| \geq 4$, and the subgraph of $\Gamma(R)$ spanned by $(M^2 - \{0\}) \cup (M - M^2)$ contains a $K_{3,3}$. The same conclusion holds if $|M^2| = 5$.

So the proof splits into cases, as $|M^2| = 2$ or 3.

Case 1. $|M^2| = 3$.

Pick $x \in M - M^2$, and consider $(0 : x)$. Certainly, $M^2 \subseteq (0 : x) \subseteq M$, and so 3 divides $|(0 : x)|$. If $|(0 : x)| = 3$, then Anderson and Nasseer (see [2, p. 512]) prove that $R$ is isomorphic to one of $\mathbb{Z}_{27}, \mathbb{Z}_3[X]/(X^3), \mathbb{Z}_9[X]/(X^2 - 3, 3X), \mathbb{Z}_9[X]/(X^2 - 6, 3X)$. Also note that each of these rings has a planar zero-divisor graph whose clique number is 3.

$|0 : x| = 6$ is impossible, as each $M^i/M^{i+1}$ is an $R/M$-vector space.
If $|(0 : x)| \geq 9$, then
$$|(0 : x) - (\{x\} \cup M^2)| \geq 5;$$
so pick (nonzero) $x_1, x_2, x_3 \in ((0 : x) - (\{x\} \cup M^2))$. Then the subgraph of $\Gamma(R)$
spanned by
$$\{x_1, x_2, x_3, x\} \cup (M^2 - \{0\})$$
contains a $K_{3,3}$.

Thus, if $|M^2| = 3$, then either $R$ is one of the rings listed above, or $\Gamma(R)$ is
nonplanar.

Case 2. $|M^2| = 2$.

As in [2, p. 513], we may pick $x \in M - M^2$ with the property that $x^2 = 0$.
Thus $Rx - \{0\}$ is a clique, and consequently $|Rx| \leq 5$. Since $|M^2| = |R/M| = 2$
(and thus $|R|$ is even), we see that $|Rx| = 2$ or 4. In [2, p. 513], Anderson and
Nasseer show that $|Rx| = 2$ is impossible. Thus $|Rx| = 4$. Since $x^2 = 0$, we have
$Rx \subset (0 : x)$. So either $|(0 : x)| = 4$ or $|(0 : x)| \geq 8$. However, in the latter case,
since one may pick 3 nonzero elements of $Rx$ and 3 elements of $(0 : x) - Rx$, $\Gamma(R)$
will contain a $K_{3,3}$.

So it remains only to consider the case where $|(0 : x)| = 4$. Since $Rx \subset (0 : x)$,
and $|Rx| = 4 = |(0 : x)|$, we must have $Rx = (0 : x)$. Now $M^2$ contains one
non-zero element; for the remainder of this proof let us denote this element by $y$.
Observe that since $M^3 = 0$, $y \in (0 : x)$; so that $(0 : x) = \{0, y, x, x + y\}$ (recall
$x \notin M^2$, so $x \neq y$).

Note that since $|Rx| = |R|/(0 : x)|$ and $|R/M| = |M^2| = 2$, we must have
$|R| = 16$ and $|M| = 8$. So pick $m \in M - (0 : x)$, and note that $M$ is generated
by $x$ and $m$. To see this, observe that $x + m \neq 0$, else $m = -x \in (0 : x)$, and
thus $x + m \in M - (0 : x)$; similarly $y + m, x + y + m \in M - (0 : x)$. Hence
$M = \{0, y, x, x + y, m, m + x, m + y, m + x + y\}$.

Observe that $xm \in M^2$, but $xm \neq 0$, as $m \notin (0 : x)$. So $xm = y$. Also note that
$m^2 = 0$ or $m^2 = y = xm$.

Since $|R| = 16$, the proof now splits into cases, as $char(R)$ equals 16, 8, 4, or 2.

If $char(R) = 16$, then $R \cong \mathbb{Z}_{16}$. Again, note that this ring has a planar zero-
divisor graph with clique number 3, but $M^3 \neq 0$.

If $char(R) = 2$, then Anderson and Nasseer [2, p. 513] have shown that
$R \cong \mathbb{Z}_2[X, Y]/(X^2, Y^2)$ or $R \cong \mathbb{Z}_2[X, Y]/(X^2, Y^2 - XY)$, as $m^2 = 0$ or $m^2 = xm$,
respectively. Again, both of these rings have a planar zero-divisor graph whose
clique number is 3.

If $char(R) = 8$, note that $2 \in M - M^2$ since $2^2 = 4$, but $2 \in M^2$ would imply
that $2^2 = 0$. Then $M$ is generated by 2 and $x$ (recall $x$ was chosen in $M - M^2$ with
$x^2 = 0$). Then $2x = 4$ or 0, and if $2x = 4$, in [2, p. 513] it is shown that
$R \cong \mathbb{Z}_8[X]/(2X - 4, X^2)$ (whose zero-divisor graph is planar with clique number
3). If $2x = 0$, one notes that
$$ann(2) = ann(6) = ann(x + 2) \supseteq \{0, 4, x, x + 4\},$$
which gives rise to a $K_{3,3}$ in $\Gamma(R)$. In fact, it is not too hard to see that in this case
$R \cong \mathbb{Z}_8[X]/(2X, X^2)$. To see this, take $\phi : \mathbb{Z}_8[X] \to R$ by $\phi(X) = x$. One notes
that $\phi$ is onto and that $ker(\phi) = (2X, X^2)$.

Lastly, if $char(R) = 4$, there are many cases to consider. Certainly $2 \in M$.
So either $2 \in M^2$ or $2 \notin M^2$. If $2 \in M^2$, then Anderson and Nasseer [2, pp.
513-514] show that $R$ is isomorphic to either $\mathbb{Z}_4[X,Y]/(X^2,XY - 2,Y^2,2X,2Y)$ or $\mathbb{Z}_4[X,Y]/(X^2,XY - 2,Y^2 - XY,2X,2Y)$. Both of these rings have a planar zero-divisor graph whose clique number is 3.

If $2 \notin M^2$, as before we may choose $m$ so that $M = (2,m)$ (where $m \in M - \text{ann}(2)$). If $2m \neq 0$, Anderson and Nasseer [2, p. 513] show $R$ is isomorphic to $\mathbb{Z}_4[X]/(X^2)$ or $\mathbb{Z}_4[X]/(X^2 - 2X)$, as $m^2 = 0$ or $m^2 \in (M^2 - \{0\}) = \{2m\}$. Once again, each of these rings has a planar zero-divisor graph whose clique number is 3.

So we are left to consider the case where $2m = 0$. To refresh the reader’s memory, we have $m$ so that $M = (2,m)$ and $M^2 - \{0\} = \{y\}$.

Case 1. $2m = 0$ and $m^2 = 0$.

In this case, note that

$$\text{ann}(2) = \{0,2,m,2 + m, y, y + 2, y + m, y + m + 2\} = M.$$ 

Observe that $2^2 = 0$ since $\text{char}(R) = 4$, $2m = 0$ by hypothesis, and $2y \in M^3 = 0$. Similarly, $\text{ann}(y) = \text{ann}(m) = M$, and thus the subgraph of $\Gamma(R)$ spanned by $\{2,y,m,2 + m, y, y + m\}$ contains a $K_{3,3}$.

Case 2. $2m = 0$ and $m^2 \neq 0$ (i.e., $m^2 = y$).

Here, just as above, one checks that $\text{ann}(y) = \text{ann}(2) = \text{ann}(y + 2) = M$, again giving rise to a $K_{3,3}$ in $\Gamma(R)$, and the proof is complete.

Our work up to this point proves the following theorem,

**Theorem 3.7.** Let $R$ be a finite commutative ring (not a field), and $k$ a (finite) field. Then $\Gamma(R)$ is a planar graph if and only if $R$ is isomorphic to one of the following 44 types of rings:

- $\mathbb{Z}_2 \times k$, $\mathbb{Z}_2 \times k$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$,
- $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_2[X]/(X^2)$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$,
- $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^3)$, $\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2X,X^2 - 2)$, $\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(X^2)$,
- $\mathbb{Z}_4 \times \mathbb{Z}_2[X]/(X^2)$, $\mathbb{Z}_9$, $\mathbb{Z}_9[X]/(X^2)$,
- $\mathbb{Z}_9$, $\mathbb{Z}_9[X]/(X^3)$, $\mathbb{Z}_4[X]/(2X,X^2 - 2)$, $\mathbb{Z}_16$, $\mathbb{Z}_2[X]/(X^4)$,
- $\mathbb{Z}_4[X]/(2X,X^3 - 2)$, $\mathbb{Z}_4[X]/(X^2 - 2)$, $\mathbb{Z}_4[X]/(X^2 + 2X + 2)$, $\mathbb{Z}_4[X]/(X^2)$,
- $\mathbb{Z}_4[X]/(X^2+X+1)$, $\mathbb{Z}_2[X,Y]/(X,Y)^2$, $\mathbb{Z}_4[X]/(2X)^2$,
- $\mathbb{Z}_{27}$, $\mathbb{Z}_{27}[X]/(X^2)$, $\mathbb{Z}_9[X]/(X^2 - 3X)$, $\mathbb{Z}_9[X]/(X^2 - 6X)$,
- $\mathbb{Z}_2[X,Y]/(X^2,Y^2 - XY)$, $\mathbb{Z}_2[X,Y]/(X,Y)^2$, $\mathbb{Z}_8[X]/(2X-4, X^2)$,
- $\mathbb{Z}_4[X]/(X^2)$, $\mathbb{Z}_4[X]/(X^2 - 2X)$,
- $\mathbb{Z}_4[X,Y]/(X^2,XY - 2,Y^2 - XY,2X,2Y)$, $\mathbb{Z}_4[X,Y]/(X^2,XY - 2,Y^2,2X,2Y)$,
- $\mathbb{Z}_{20}$, or $\mathbb{Z}_{20}[X]/(X^2)$.

To finish, we state a corollary and some remarks which are immediate, yet interesting in their own right.

**Corollary 3.8.** Let $R$ be a finite, commutative local ring (not a field) with either $|R| \geq 28$ or $|\mathbb{Z}(R)| \geq 10$. Then $\Gamma(R)$ is not planar.

**Proof.** One need only examine the previous list of rings. 

**Corollary 3.8** is best possible since $|\mathbb{Z}_{27}| = 27$ and $|\mathbb{Z}(\mathbb{Z}_{27})| = 9$. Also note that all finite commutative rings $R$ with $|\Gamma(R)| \leq 4$ have planar zero-divisor graphs (since $\Gamma(R)$ can not contain a $K_5$ or a $K_{3,3}$), and each such graph can be realized by a local ring when $|\Gamma(R)| \leq 3$ (cf. [3, Example 2.1]). As a consequence of [4,
Theorem 2.10], one can see that if $|\Gamma(R)| = 5$, then $\Gamma(R)$ is planar. Further, it is shown that $\Gamma(R)$ cannot be realized by a local ring (for then $M$ would have 6 elements). And, one can easily find examples of finite commutative rings $R$ with $6 \leq |\Gamma(R)| \leq 8$ having both planar and non-planar zero-divisor graphs. In this case, such planar zero-divisor graphs can be realized by local rings only when $|\Gamma(R)| = 7$ or 8. Also, observe that a local ring $R$ (not a field) having a planar zero-divisor graph necessarily has 4, 8, 16, 9, 27, or 25 elements (and $|\Gamma(R)| = 1, 3, 7, 2, 8,$ or 4, respectively).

It is well-known that not all connected graphs on a given number of vertices can be realized as $\Gamma(R)$ for some appropriate $R$. For instance, an octagon cannot be $\Gamma(R)$, since in [4, Theorem 2.3], it is shown that any two vertices of $\Gamma(R)$ can be joined by a path of length at most 3. However, the previous theorem lets us find examples of graphs that satisfy this requirement, yet cannot be realized as $\Gamma(R)$. For example, consider the graph obtained by pasting together two copies of $K_4$ at a ‘corner’. In this graph, any two vertices can be joined by a path of length 2, and this graph has girth 4 (and is planar!), but is not one of the graphs arising from the previous list, and hence cannot be realized as $\Gamma(R)$ for any commutative ring $R$.

As a final remark, $\Gamma(R)$ can be an infinite planar graph; for example, $\Gamma(\mathbb{Z} \times \mathbb{Z}_2)$ is a planar graph. This prompts the question as to which infinite commutative rings $R$ have planar zero-divisor graphs. At this point, this is an open question.

References