Homework #1

8.10. a) The posterior is $\pi(\lambda|x) = \text{gamma} \left( n(\bar{x} + \alpha - 1) + 1, \frac{\beta}{n(1+\beta)} \right)$. Thus

$$P(H_0) = P(\lambda \leq \lambda_0) = \int_0^{\lambda_0} \pi(\lambda|x)dx$$

and

$$P(H_1) = 1 - P(H_0).$$

b) As $\pi(\lambda|x) = \text{gamma} \left( n \left( \bar{x} + \frac{3}{2} \right) + 1, \frac{2}{3n} \right)$, we have

$$3n\lambda \sim \text{gamma} \left( \frac{2n\bar{x} + 3n + 2}{2}, \frac{1}{2} \right) = \chi^2_{2n\bar{x}+3n+2}.$$

Use $\chi^2$ table, we can calculate $P(H_0)$ and $P(H_1)$.

8.11. a) The posterior is $\text{IG} \left( \alpha + \frac{n-1}{2}, \left( \frac{(n-1)S^2}{2} + \frac{1}{\beta} \right)^{-1} \right)$. Denote

$$a = \alpha + \frac{n-1}{2} \quad \text{and} \quad b = \left( \frac{(n-1)S^2}{2} + \frac{1}{\beta} \right)^{-1}.$$

Then

$$P(\sigma^2 \leq 1|S^2) = \int_0^1 \frac{1}{\Gamma(a)b^a} \frac{1}{x^{a+1}} e^{-\frac{1}{b}x} dx = \frac{1}{\Gamma(a)} \int_0^\infty y^{a-1} e^{-y} dy$$

is increasing in $b$; and

$$P(\sigma^2 > 1|S^2) = \frac{1}{\Gamma(a)} \int_0^1 y^{a-1} e^{-y} dy$$

is decreasing in $b$. Further, $b$ is decreasing in $S^2$. Thus the acceptance region of the Bayes test is

$$S^2 \leq k$$

for some specific $k$.

b) Since it has the same form as the LRT, it is possible to choose suitable prior parameter such that

$$P \left( P(\sigma^2 \leq 1|S^2) > P(\sigma^2 > 1|S^2) \right) = 1 - \alpha.$$

So the LRT coincides with the Bayes test.