Review for final
Ch3 (3.4), Ch4, ch5 (5.1), (5.2)

1. If \( f : E \to \mathbb{R} \) unif. conti. and \( E \) is bounded, then \( f(E) \) is bounded.

2. If \( E \) is compact and \( \{x_n\} \subset E \), then \( \exists x_0 \in E \), \( \exists \) subsequence \( \{x_{n_k}\} \) such that \( x_{n_k} \to x_0 \).

3. If \( f : E \to \mathbb{R} \) is continuous and \( E \) is compact, then \( f(E) \) is compact.

4. If \( f : E \to \mathbb{R} \) is 1-1 and conti. with \( E \) compact, then \( f^{-1} : f(E) \to \mathbb{R} \) is conti.

5. (Bolzano) Suppose \( f : [a,b] \to \mathbb{R} \) is conti. and \( f(a)f(b) < 0 \). Then, \( \exists c \in (a,b), f(c) = 0 \).

- Let \( f : A \to \mathbb{R} \) be conti. with \( A \) connected. Suppose \( a, b \in A \) and \( y \) is between \( f(a) \) and \( f(b) \), then \( \exists c \in (a,b), f(c) = y \).

6. If \( f : [a,b] \to \mathbb{R} \) is conti., then \( \exists c, d \) such that \( f([a,b]) = [c,d] \).

7. Let \( f : A \to \mathbb{R} \) be conti. and 1-1 with \( A \) connected. Then, \( f \) is monotone.

8. \( f : D \to \mathbb{R} \) is differentiable at \( x_0 \in D \) if

\[
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \text{ exists.}
\]

- \( f'(x_0) = \lim_{t \to 0} \frac{f(x_0 + t) - f(x_0)}{t} \).

- If \( f \) is differentiable at \( x_0 \), then \( f \) is continuous at \( x_0 \).

9. If \( f, g : D \to \mathbb{R} \) are differentiable at \( x \), so are \( f + g, fg \) and \( f/g \) (if \( g(x) \neq 0 \))

\[
(f + g)'(x) = f'(x) + g'(x),
\]

\[
(fg)'(x) = f'(x)g(x) + f(x)g'(x),
\]

and

\[
(f/g)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.
\]
10. (Chain rule) Suppose \( f : D \to \mathbb{R} \) and \( g : f(D) \to \mathbb{R} \) are differentiable at \( x_0 \) and \( f(x_0) \) respectively. Then, \( g \circ f \) is differentiable at \( x_0 \) and 
\[
(g \circ f)'(x_0) = g'(f(x_0))f'(x_0).
\]

11. For any \( \alpha \in \mathbb{Q} \), we have 
\[
(x^\alpha)' = \alpha x^{\alpha-1}.
\]

12. \( f : D \to \mathbb{R} \) has a relative maximum at \( x_0 \in D \) if \( \exists \) neighborhood \( Q \) of \( x_0 \) such that 
\[
f(x) \leq f(x_0), \quad \forall x \in Q \cap D.
\]

- Suppose \( f : [a, b] \to \mathbb{R} \) has a relative maximum or minimum at \( x_0 \in (a, b) \). If \( f \) is differentiable at \( x_0 \), then \( f'(x_0) = 0 \).

13. (Rolle) Suppose \( f : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\) and differentiable on \((a, b)\). If \( f(a) = f(b) \), then \( \exists \ c \in (a, b), \ f'(c) = 0 \).

- (MVT) Suppose \( f : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\) and differentiable on \((a, b)\), 
\[
\frac{f(b) - f(a)}{b - a} = f'(c).
\]

14. Suppose \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\).

- (a) If \( f'(x) \neq 0, \ \forall \ x \), then \( f \) is 1-1
- (b) If \( f'(x) = 0, \ \forall \ x \), then \( f \) is constant
- (c) If \( f'(x) > 0, \ \forall \ x \), then \( f \) is strictly increasing

15. Suppose \( f \) is differentiable on \([a, b]\) and \( \lambda \) is between \( f'(a) \) and \( f'(b) \). Then, \( \exists \ c \in (a, b), \ f'(c) = \lambda \).

16. Suppose \( f, g \) are conti. on \([a, b]\) and differentiable on \((a, b)\), then \( \exists \ c \in (a, b) \)
\[
\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.
\]
17. (L'Hospital’s rule) Suppose $f$ and $g$ are differentiable on $[a, b]$. If $x_0 \in [a, b]$ such that

(a) $g'(x) \neq 0, \forall x$
(b) $f(x_0) = g(x_0) = 0$
(c) $f'/g'$ has a limit at $x_0$,

then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$ 

18. If $f : [a, b] \to \mathbb{R}$ is differentiable with $f'(x) \neq 0, \forall x$, then $f$ is 1-1, $f^{-1}$ is differentiable and

$$(f^{-1})'(f(x)) = \frac{1}{f''(x)}, \quad \forall x \in [a, b].$$

19. $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$ if $a = x_0 < x_1 < \cdots < x_n = b$.

- $Q$ is a refinement of $P$ if $P \subset Q$.

20. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ and $f : [a, b] \to \mathbb{R}$. Define

$$M_i(f) = \sup \{f(x) : x \in [x_{i-1}, x_i]\}$$

and

$$m_i(f) = \inf \{f(x) : x \in [x_{i-1}, x_i]\}.$$ 

Then the upper sum is

$$U(P, f) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1})$$

and the lower sum is

$$L(P, f) = \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}).$$

Define upper and lower integrals as

$$\int_{a}^{b} f dx = \inf_{P} U(P, f) \quad \text{and} \quad \int_{a}^{b} f dx = \sup_{P} L(P, f).$$
\( f \text{ is integrable iff } \int_a^b f \, dx = \int_a^b f \, dx. \)

21. Let \( f : [a, b] \rightarrow \mathbb{R} \) be bounded.

(a) If \( P \subset Q \), then \( L(P, f) \leq L(Q, f) \) and \( U(P, f) \geq U(Q, f) \).

(b) \( L(P, f) \leq U(Q, f) \).

(c) \( \int_a^b f \, dx \leq \int_a^b f \, dx. \)

22. \( f \in R(x) \) iff \( \forall \epsilon > 0, \exists P \) such that
\( U(P, f) - L(P, f) < \epsilon. \)

23. If \( f : [a, b] \rightarrow \mathbb{R} \) is monotone, then \( f \in R(x) \).

24. If \( f : [a, b] \rightarrow \mathbb{R} \) is continuous, then \( f \in R(x) \).