Review for the second midterm  
Ch2, Ch3 (3.1, 3.2, 3.3)

1. \( f : D \to \mathbb{R} \) has a limit \( L \) at \( x_0 \) iff \( \forall \epsilon > 0, \exists \delta > 0, \forall 0 < |x - x_0| < \delta \), we have \( |f(x) - L| < \epsilon \).

- \( f \) has a limit at \( x_0 \) iff for each sequence \( \{x_n\} \subset D \setminus \{x_0\} \) converging to \( x_0 \), \( \{f(x_n)\} \) is Cauchy
- If \( f \) has a limit at \( x_0 \), then \( \exists \) neighborhood \( Q \) of \( x_0 \), \( \exists M \in \mathbb{R} \) such that \( |f(x)| \leq M, \forall x \in Q \cap D \).

2. If \( f, g : D \to \mathbb{R} \) have limits \( L_1, L_2 \) at \( x_0 \), then
   (a) \( f + g, fg \) have limits \( L_1 + L_2, L_1L_2 \) at \( x_0 \),
   (b) \( f/g \) has limit \( L_1/L_2 \) provided \( L_2 \neq 0 \),
   (c) If \( f(x) \leq g(x), \forall x \in D \), then \( L_1 \leq L_2 \).

- If \( f : D \to \mathbb{R} \) is bounded in a neighborhood of \( x_0 \), and \( g : D \to \mathbb{R} \) has limit 0 at \( x_0 \), then \( fg \) has limit 0 at \( x_0 \).

3. \( f \) is monotone iff \( f \) is either increasing or decreasing

- Let \( f : [\alpha, \beta] \to \mathbb{R} \) be monotone. Then
  (a)
  \[ J = \{x \in (\alpha, \beta) : f \text{ does not have a limit at } x \} \]
  is countable,
  (b) \( \forall x_0 \not\in J, \lim_{x \to x_0} f(x) = f(x_0) \).

4. \( f : D \to \mathbb{R} \) is continuous at \( x_0 \in D \) iff \( \forall \epsilon > 0, \exists \delta > 0, \forall |x - x_0| < \delta \), we have \( |f(x) - f(x_0)| < \epsilon \).

5. The following statements are equivalent:
   (a) \( f \) is continuous at \( x_0 \),
   (b) \( f \) has a limit \( f(x_0) \) at \( x_0 \),
   (c) For every sequence \( \{x_n\} \subset D \) converging to \( x_0 \), the sequence \( \{f(x_n)\} \) converges to \( f(x_0) \).
6. If \( f, g : D \to \mathbb{R} \) are continuous at \( x_0 \in D \), then
   
   (a) \( f + g, fg \) are continuous at \( x_0 \),
   
   (b) \( f/g \) is continuous at \( x_0 \) provided \( g(x_0) \neq 0 \).

7. If \( f : D \to \mathbb{R} \) and \( g : D' \to \mathbb{R} \) with \( \text{im}(f) \subset D' \) where \( f \) is continuous at \( x_0 \) and \( g \) is continuous at \( f(x_0) \), then \( g \circ f : D \to \mathbb{R} \) is continuous at \( x_0 \).

8. \( f : D \to \mathbb{R} \) is uniformly continuous on \( E \subset D \) iff \( \forall \epsilon > 0, \exists \delta > 0, \forall x, y \in E, |x - y| < \delta, \) we have \( |f(x) - f(y)| < \epsilon \).

   • If \( f : D \to \mathbb{R} \) is uniformly continuous and \( x_0 \) is an accumulation point of \( D \), then \( f \) has a limit at \( x_0 \).

9. \( E \subset \mathbb{R} \)
   
   • \( E \) closed if it contains all its accumulation points.
   
   • \( E \) open if \( \forall x \in E, \exists \) neighborhood of \( Q \) of \( x, Q \subset E \).
   
   • \( E \) compact if every open cover has a finite subcover.
   
   • \( E \) closed iff \( \mathbb{R} \setminus E \) open
   
   • \( E \) compact iff \( E \) is closed and bounded.

10. If \( f : D \to \mathbb{R} \) is continuous and \( D \) is compact, then \( f \) is uniformly continuous.