1. Let \( f : [0, 2] \to \mathbb{R} \) be given by \( f(x) = \frac{x}{1+x} \). Use \( \epsilon-\delta \) argument, prove that \( f \) has a limit at \( x = 1 \).

Solution: Guess \( L = \frac{1}{2} \).

\[
\forall \epsilon > 0, \exists \delta = 2\epsilon > 0, \forall 0 < |x - 1| < \delta, \text{ we have }
\]

\[
\left| \frac{x}{1+x} - \frac{1}{2} \right| = \frac{|2x - 1 - x|}{2(1 + x)} = \frac{|x - 1|}{2(1 + x)} < \frac{\delta}{2} = \epsilon.
\]

So, \( f \) has limit \( \frac{1}{2} \) at \( x = 1 \).

2. Using \( \epsilon-\delta \) argument, prove that the function

\[
f(x) = \begin{cases} 
2x \sin \frac{1}{x}, & x \neq 0 \\
0, & x = 0
\end{cases}
\]

is continuous at 0.

Proof: \( \forall \epsilon > 0, \exists \delta = \frac{\epsilon}{2} > 0, \forall |x| < \delta, \text{ we have } \)

\[
|f(x) - f(0)| = \left| 2x \sin \frac{1}{x} - 0 \right| \leq 2|x| < 2\delta = \epsilon.
\]

Thus, \( f \) is continuous at 0.
3. Using algebra of limits, conclude that the function
\[ g(x) = \frac{x\sqrt{1 + x} - x}{x^2 \cos x}, \quad x \neq 0, \]
has limit at 0 and find the limit. (You can use the fact that \( \lim_{x \to 0} \cos x = 1 \)).
Solution:

\[
\lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{x\sqrt{1 + x} - x}{x^2 \cos x} \lim_{x \to 0} \frac{1}{x^2 \cos x}
= \lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x}
= \lim_{x \to 0} \frac{(\sqrt{1 + x} - 1)(\sqrt{1 + x} + 1)}{x(\sqrt{1 + x} + 1)}
= \lim_{x \to 0} \frac{1 + x - 1}{x(\sqrt{1 + x} + 1)}
= \lim_{x \to 0} \frac{1}{\sqrt{1 + x} + 1}
= \frac{1}{2}.
\]

4. Let \( f : (0, 1) \to \mathbb{R} \) be continuous at \( x_0 \in (0, 1) \), and \( f(x_0) = -1 \). Using \( \epsilon-\delta \) argument, show that \( \exists \, \delta > 0 \) s.t. \( (x_0 - \delta, x_0 + \delta) \subseteq (0, 1) \) and \( f(x) < -\frac{1}{2} \) for all \( x \in (x_0 - \delta, x_0 + \delta) \).

Proof: Choose \( \delta \) such that \( x_0 - \delta \geq 0 \) and \( x_0 + \delta \leq 1 \). Namely,
\[
\delta \leq x_0, \quad \delta \leq 1 - x_0.
\]
Since \( f \) is continuous at \( x_0 \in (0, 1) \), \( \forall \, \epsilon > 0, \exists \, \delta_1, \forall \, |x - x_0| < \delta_1 \), we have
\[
|f(x) - (-1)| < \epsilon.
\]
Take \( \epsilon = \frac{1}{2} \) and \( \delta = \min(x_0, 1 - x_0, \delta_1) \). Then
\[
(x_0 - \delta, x_0 + \delta) \subseteq (0, 1)
\]
and for \( x \in (x_0 - \delta, x_0 + \delta) \), we have
\[
f(x) < -1 + \frac{1}{2} = -\frac{1}{2}.
\]
5. Let \( f, g : \mathbb{R} \to \mathbb{R} \) be bounded and uniformly continuous functions on \( \mathbb{R} \). Prove that \( h = fg \) (product of \( f \) and \( g \)) is uniformly continuous on \( \mathbb{R} \).

Proof: Since \( f \) is bounded, \( \exists M_1 > 0 \) such that
\[
|f(x)| \leq M_1, \quad \forall x \in \mathbb{R}.
\]
Similarly, \( \exists M_2 > 0 \) such that
\[
|g(x)| \leq M_2, \quad \forall x \in \mathbb{R}.
\]
Let \( M = \max(M_1, M_2) \).

Since \( f \) is uniformly continuous, \( \forall \epsilon > 0, \exists \delta_1 > 0, \forall |x - y| < \delta_1, \) we have
\[
|f(x) - f(y)| < \frac{\epsilon}{2M}.
\]
Similarly, \( \exists \delta_2 > 0, \forall |x - y| < \delta_2, \) we have
\[
|g(x) - g(y)| < \frac{\epsilon}{2M}.
\]
Let \( \delta = \min(\delta_1, \delta_2) \). For \( |x - y| < \delta \), we have
\[
|h(x) - h(y)| = |f(x)g(x) - f(y)g(y)|
\leq |f(x)g(x) - f(y)g(x)| + |f(y)g(x) - f(y)g(y)|
\leq |f(x) - f(y)||g(x)| + |f(y)||g(x) - g(y)|
< \frac{\epsilon}{2M}M + M\frac{\epsilon}{2M}
= \epsilon.
\]