41. Find an interval of length 1 than contains a root of the equation $xe^x = 1$.
Solution: Let $f(x) = xe^x - 1$. Then $f(0) = -1 < 0$ and $f(1) = e - 1 > 0$. There is $c \in (0, 1)$ such that $f(c) = 0$. The answer is $(0, 1)$.

42. Find an interval of length 1 than contains a root of the equation $x^3 - 6x^2 + 2.826 = 0$.
Solution: Let $f(x) = x^3 - 6x^2 + 2.826$. Then $f(0) = 2.826 > 0$ and $f(1) = 1 - 6 + 2.826 < 0$. There is $c \in (0, 1)$ such that $f(c) = 0$. The answer is $(0, 1)$.

44. Suppose that $f : [a, b] \to [a, b]$ is continuous. Prove that there is at least one fixed point in $[a, b]$, i.e. there is $x$ such that $f(x) = x$.
Proof: Let $g(x) = f(x) - x$. Then
\[ g(a) = f(a) - a \geq a - a = 0 \]
and
\[ g(b) = f(b) - b \leq b - b = 0. \]
If $g(a) = 0$ or $g(b) = 0$, the conclusion is obvious with $x = a$ or $b$. If $g(a) > 0$ and $g(b) < 0$, by Bolzano’s theorem, there exists $x \in (a, b)$ such that $g(x) = 0$, and hence,
\[ f(x) = x. \]

45. If $f : [a, b] \to \mathbb{R}$ is 1-1 and has the intermediate-value property—that is, if $y$ is between $f(u)$ and $f(v)$, there is $x$ between $u$ and $v$ such that $f(x) = y$—show that $f$ is continuous. (Hint: First show that $f$ is monotone.)
Proof: Suppose $f$ is not monotone. Then
1. $\exists x < y < z$, $x, y, z \in [a, b]$ such that $f(x) < f(y)$ and $f(z) < f(y)$, or
2. $\exists x < y < z$, $x, y, z \in [a, b]$ such that $f(x) > f(y)$ and $f(z) > f(y)$.

For case 1, let us suppose $f(x) < f(z) < f(y)$. By the IVP, $\exists c \in (x, y)$ such that $f(c) = f(z)$. This contradict from $f$ being 1 − 1. Other cases can be treated similarly. Thus, $f$ is monotone.
Without loss of generality, we may and will assume that $f$ is increasing. If $f$ is not continuous at a point $x_0 \in (a,b)$, then by a theorem,

$$\sup\{f(x) : x < x_0\} < \inf\{f(y) : y > x_0\}.$$  

Let $L$ be such that $L \neq f(x_0)$ and

$$\sup\{f(x) : x < x_0\} < L < \inf\{f(y) : y > x_0\}.$$  

Taking $u < x_0$ and $v > x_0$. Then

$$f(u) < L < f(v)$$

and there is no point $c$ such that $f(c) = L$; otherwise, $\sup\{f(x) : x < x_0\} < L$ implies $c \geq x_0$, and $L < \inf\{f(y) : y > x_0\}$ implies $c \leq x_0$. Thus, $c = x_0$. However, $L \neq f(x_0)$. This contradiction implies the continuity of $f$.  

1. Let $(x_0, y_0)$ be an arbitrary point on the graph of the function $f(x) = x^2$. For $x_0 \neq 0$, find the equation of the line tangent to the graph of $f$ at that point by finding a line that intersects the curve in exactly one point. Do not use the derivative to find this line.

Solution: Let the line by

$$y = y_0 + k(x - x_0).$$

As $y_0 = x_0^2$, the intersection is

$$x^2 = x_0^2 + k(x - x_0).$$

Namely,

$$x^2 - kx + (k - x_0)x_0 = 0$$

has exactly one solution. Thus

$$(-k)^2 - 4(k - x_0)x_0 = 0.$$  

Therefore

$$k^2 - 4x_0k + 4x_0^2 = 0.$$  

Thus

$$k = 2x_0$$
and the line is 

\[ y = y_0 + 2x_0^2(x - x_0). \]

3. Use the definition to find the derivative of \( f(x) = \sqrt{x} \), for \( x > 0 \). Is \( f \) differentiable at zero? Explain.
Solution: For \( x > 0 \),

\[
\begin{align*}
    f'(x) &= \lim_{t \to 0} \frac{f(x + t) - f(x)}{t} \\
    &= \lim_{t \to 0} \frac{\sqrt{x + t} - \sqrt{x}}{t} \\
    &= \lim_{t \to 0} \frac{(\sqrt{x + t} - \sqrt{x})(\sqrt{x + t} + \sqrt{x})}{t(\sqrt{x + t} + \sqrt{x})} \\
    &= \lim_{t \to 0} \frac{t}{t(\sqrt{x + t} + \sqrt{x})} \\
    &= \frac{1}{\sqrt{x + x}} = \frac{1}{2\sqrt{x}}.
\end{align*}
\]

It is not differentiable because 

\[
\frac{f(t) - f(0)}{t} = \frac{\sqrt{t}}{t} = \frac{1}{\sqrt{t}}
\]

tends to \( \infty \).

6. Suppose \( f : (a, b) \to \mathbb{R} \) is differentiable at \( x \in (a, b) \). Prove that 

\[
\lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h}
\]

exists and equals \( f'(x) \). Give an example of a function where this limit exists, but the function is not differentiable.
Solution:

\[
\begin{align*}
    \lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h} &= \lim_{h \to 0} \frac{f(x + h) - f(x) + f(x) - f(x - h)}{2h} \\
    &= \frac{1}{2} \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} + \frac{1}{2} \lim_{h \to 0} \frac{f(x - h) - f(x)}{-h} \\
    &= \frac{1}{2} f'(x) + \frac{1}{2} f'(x) = f'(x).
\end{align*}
\]
Let $f(x) = |x|$. Then $f$ is not differentiable at $x = 0$.
However
\[
\frac{f(h) - f(-h)}{2h} = \frac{|h| - |-h|}{2h} = 0
\]
is convergent.

8. A function $f : (a, b) \to \mathbb{R}$ is said to be uniformly differentiable iff $f$ is differentiable on $(a, b)$ and for each $\epsilon > 0$, there is $\delta > 0$ such that $0 < |x - y| < \delta$ and $x, y \in (a, b)$ imply that
\[
\left| \frac{f(x) - f(y)}{x - y} - f'(x) \right| < \epsilon.
\]
Prove that if $f$ is uniformly differentiable on $(a, b)$, then $f'$ is continuous on $(a, b)$.

Proof: Since $f$ is uniformly differentiable on $(a, b)$, for each $\epsilon > 0$, there is $\delta > 0$ such that $0 < |x - y| < \delta$ and $x, y \in (a, b)$ imply that
\[
\left| \frac{f(x) - f(y)}{x - y} - f'(x) \right| < \frac{\epsilon}{2}.
\]
Interchange $x$ and $y$, we have
\[
\left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \frac{\epsilon}{2}.
\]
Thus, for each $\epsilon > 0$, there is $\delta > 0$ such that $0 < |x - y| < \delta$ and $x, y \in (a, b)$ imply that
\[
|f'(x) - f'(y)| = \left| f'(x) - \frac{f(x) - f(y)}{x - y} + f'(y) \frac{f(x) - f(y)}{x - y} \right|
\leq \left| \frac{f(x) - f(y)}{x - y} - f'(x) \right| + \left| f'(y) \frac{f(x) - f(y)}{x - y} - f'(y) \right|
< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]
\[\blacksquare\]