11. Suppose $f$, $g$, and $h : D \to \mathbb{R}$ where $x_0$ is an accumulation point of $D$, $f(x) \leq g(x) \leq h(x)$ for all $x \in D$, and $f$ and $h$ have limits at $x_0$ with $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x)$. Prove that $g$ has a limit at $x_0$ and

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x).$$

Proof: Denote

$$L = \lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x).$$

Then, $\forall \, \epsilon > 0$, $\exists \, \delta_1 > 0$, when $0 < |x - x_0| < \delta_1$, we have

$$|f(x) - L| < \epsilon$$

and hence

$$L - \epsilon < f(x) < L + \epsilon.$$

Similarly, $\forall \, \epsilon > 0$, $\exists \, \delta_2 > 0$, when $0 < |x - x_0| < \delta_2$, we have

$$L - \epsilon < h(x) < L + \epsilon.$$

Let $\delta = \min(\delta - 1, \delta_2)$. When $0 < |x - x_0| < \delta$, we have

$$L - \epsilon < f(x) \leq g(x) \leq h(x) < L + \epsilon.$$

namely

$$|g(x) - L| < \epsilon.$$

Thus

$$\lim_{x \to x_0} g(x) = L.$$

12. Suppose $f : D \to \mathbb{R}$ has a limit at $x_0$. Prove that $|f| : D \to \mathbb{R}$ has a limit at $x_0$ and that

$$\lim_{x \to x_0} |f(x)| = \left| \lim_{x \to x_0} f(x) \right|.$$

Proof: Denote

$$\lim_{x \to x_0} f(x) = L.$$
Then, \( \forall \epsilon > 0, \exists \delta > 0 \), when \( 0 < |x - x_0| < \delta \), we have
\[
|f(x) - L| < \epsilon.
\]
Thus,
\[
||f(x) - |L|| \leq |f(x) - L| < \epsilon.
\]
Therefore,
\[
\lim_{x \to x_0} |f(x)| = |L|.
\]

13. Define \( f: \mathbb{R} \to \mathbb{R} \) by \( f(x) = x - [x] \). Determine those points at which \( f \) has a limit, and justify your conclusion.
Solution: If \( x_0 \in \mathbb{Z} \), then there is \( n \in \mathbb{Z} \) such that \( k < x_0 < k + 1 \). For any \( \epsilon > 0 \), let \( \delta = \min(x_0 - k, k + 1 - x_0, \epsilon) \), for \( 0 < |x - x_0| < \delta \), we have
\[
|f(x) - f(x_0)| = |x - n - x_0 + n| = |x - x_0| < \delta \leq \epsilon.
\]
Thus, \( f \) has a limit at \( x_0 \).

If \( x_0 \in \mathbb{Z} \), let \( x_n = x_0 - \frac{1}{2n} \) and \( x'_n = x_0 + \frac{1}{2n} \). Then \( x_n, x'_n \to x_0 \), while
\[
f(x_n) = x_n - x_0 + 1 \to 1
\]
and
\[
f(x'_n) = x'_n - x_0 \to 0 \neq 1.
\]
Thus, \( f \) does not have a limit at \( x_0 \). Hence, \( f \) has a limit at non-integer points.

14. Define \( f: \mathbb{R} \to \mathbb{R} \) as follows:
\[
f(x) = \begin{cases} 
8x & \text{if } x \text{ is a rational number}, \\
2x^2 + 8 & \text{if } x \text{ is an irrational number}.
\end{cases}
\]
Use sequences to guess at which points \( f \) has a limit, then use \( \epsilon \)'s and \( \delta \)'s to justify your conclusion.
Solution: Take \( r_n \to x_0 \) be rational and \( q_n \to x_0 \) to be irrational. Then
\[
f(r_n) = 8r_n \to 8x_0
\]
and
\[
f(q_n) = 2q_n^2 + 8 \to 2x_0^2 + 8.
\]
For $f$ to have limit, we must have

$$8x_0 = 2x_0^2 + 8.$$ 

Hence, $x_0 = 2$ and $L = 16$. Now we prove this result.

For any $\epsilon > 0$, let $\delta = \min\left(\frac{\epsilon}{10}, 1\right)$ and $0 < |x - 2| < \delta$. If $x$ is rational, then

$$|f(x) - 16| = |8x - 16| = 8|x - 2| < 8\delta \leq \epsilon.$$ 

If $x$ is irrational, then

$$|f(x) - 16| = |2x^2 + 8 - 16| = 2|x^2 - 4| = 2|x - 2||x + 2| < 2\delta(3 + 2) = 12\delta \leq \epsilon.$$ 

Thus, $\lim_{x \to 2} f(x) = 16$. \hfill \blacksquare

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16. Define $f : (0, 1) \to \mathbb{R}$ by $f(x) = \frac{x^3 + 6x^2 + x}{x^2 - 6x}$. Prove that $f$ has a limit at 0 and find the limit.

Solution:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x^3 + 6x^2 + x}{x^2 - 6x}$$

$$= \lim_{x \to 0} \frac{x^2 + 6x + 1}{x - 6}$$

$$= -\frac{1}{6}.$$ 

\hfill \blacksquare

17. Define $f : \mathbb{R} \to \mathbb{R}$ as follows

$$f(x) = \begin{cases} 
  x - \lfloor x \rfloor & \text{if } \lfloor x \rfloor \text{ is even}, \\
  x - \lfloor x + 1 \rfloor & \text{if } \lfloor x \rfloor \text{ is odd}.
\end{cases}$$

Determine those points where $f$ has a limit, and justify your conclusion.

Solution. Suppose $x_0$ is not an integer with $\lfloor x \rfloor$ even. For $x$ near $x_0$ we have $f(x) = x - \lfloor x \rfloor$ and hence, $f$ has a limit at $x_0$ by problem 13.

Suppose $x_0$ is not an integer with $\lfloor x \rfloor$ odd. For $x$ near $x_0$ we have $f(x) = x - \lfloor x \rfloor - 1$ and hence, $f$ has a limit at $x_0$ by problem 13 and the addition property of the limit.
Suppose \( x_0 \) is an even integer, say \( 2k \). For any \( \epsilon > 0 \), let \( \delta = \min(\epsilon, 1) \). When \( 0 < |x - x_0| < \delta \), we have
\[
|f(x) - 0| = |x - 2k| < \delta \leq \epsilon.
\]
Thus, \( f \) has a limit 0 at \( x_0 \).

Suppose \( x_0 \) is an odd integer, say \( 2k + 1 \). Let \( r_n = 2k + 1 + \frac{1}{2n} \) and \( q_n = 2k + 1 - \frac{1}{2n} \). Then \( r_n, q_n \to x_0 \) while
\[
f(r_n) = r_n - (2k + 2) \to -1
\]
and
\[
f(q_n) = q_n - 2k \to 1 \neq -1.
\]
Thus \( f \) does not have a limit at all odd integers. \( \Box \)

19. Define \( f : (0, 1) \to \mathbb{R} \) by \( f(x) = \frac{\sqrt{9 - x} - 3}{x} \). Prove that \( f \) has a limit at 0 and find it.

Solution:
\[
\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sqrt{9 - x} - 3}{x}
= \lim_{x \to 0} \frac{(\sqrt{9 - x} - 3)(\sqrt{9 - x} + 3)}{x(\sqrt{9 - x} + 3)}
= \lim_{x \to 0} \frac{(9 - x - 3^2)}{x(\sqrt{9 - x} + 3)}
= \lim_{x \to 0} \frac{-1}{\sqrt{9 - x} + 3}
= \frac{-1}{3} = -\frac{1}{6}.
\]
\( \Box \)

22. Show by example that, even though \( f \) and \( g \) fail to have limits at \( x_0 \), it is possible for \( f + g \) to have a limit at \( x_0 \). Give similar examples for \( fg \) and \( f \cdot g \).

Solution. Let
\[
f(x) = [x] \text{ and } g(x) = x - [x].
\]
Then \( f \) and \( g \) fail to have limits at 0, but \( f + g = x \) has a limit at 0.

For \( fg \), we take \( f(x) = \frac{1}{[x]} \) and \( g(x) = [x] \) with \( x_0 = 1 \).

For \( f \cdot g \), we take \( f(x) = g(x) = [x] \) and \( x_0 = 1 \).