1. (18 points) Let $X$ be a continuous random variable with probability density function

$$f(x) = \frac{3}{8}x^2, \quad 0 \leq x \leq 2.$$ 

1) Find $P(0 \leq X \leq 1)$.

2) Find the expected value and the standard deviation of $X$.

3) Calculate $E(X^3)$.

**Solution.**

1) 

$$P(0 \leq X \leq 1) = \int_0^1 \frac{3}{8}x^2 \, dx = \frac{1}{8}x^3 \bigg|_0^1 = \frac{1}{8}$$

2) 

$$E(X) = \int_0^2 x \cdot \frac{3}{8}x^2 \, dx = \frac{3}{32}x^4 \bigg|_0^2 = \frac{3}{2}$$

$$E(X^2) = \int_0^2 x^2 \cdot \frac{3}{8}x^2 \, dx = \frac{3}{40}x^5 \bigg|_0^2 = \frac{12}{5}$$

$$V(X) = \frac{12}{5} - \left(\frac{3}{2}\right)^2 = \frac{3}{20}$$

$$SD(X) = \sqrt{\frac{3}{20}} = .387$$

3) 

$$E(X^3) = \int_0^2 x^3 \cdot \frac{3}{8}x^2 \, dx = \frac{3}{48}x^6 \bigg|_0^2 = 4$$
2. (18 points) Suppose $X$ has a uniform distribution on $(1, 3)$.
1) What is the probability density function of $X$?
2) What are $E(X)$ and $V(X)$ equal to?
3) Calculate $E(X^4)$.
Solution. 1) 
$$f(x) = \frac{1}{2}, \quad 1 < x < 3$$
2) 
$$E(X) = \frac{1 + 3}{2} = 2, \quad V(X) = \frac{(3 - 1)^2}{12} = \frac{1}{3}$$
3) 
$$E(X^4) = \int_{1}^{3} x^4 \frac{1}{2} dx = \left. \frac{1}{10} x^5 \right|_{1}^{3} = 24.1$$

3. (18 points) Suppose $X$ has an exponential distribution with mean 25.
1) What is the distribution function of $X$?
2) What is the variance of $X$?
3) Find $P(10 < X < 30)$.
Solution. 1) 
$$F(x) = \begin{cases} 1 - e^{-\frac{x}{25}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$
2) 
$$V(X) = \theta^2 = 25^2 = 625$$
3) 
$$P(10 < X < 30) = 1 - e^{-\frac{30}{25}} - \left( 1 - e^{-\frac{10}{25}} \right) = e^{-1.2} - e^{-1.2} = .369$$
4. (18 points) Let $X$ be a normal random variable with mean 20 and variance 25.

1) Find $P(X \leq 25)$.
2) What is the distribution of $Y = 2X + 3$? Specify its mean and variance.
3) Find a level $x_0$ such that $P(X \leq x_0) = .05$.

**Solution.** 1)

\[
P(X \leq 25) = P \left( Z \leq \frac{25 - 20}{5} \right) = P(Z \leq 1)
= .5 + .3413 = .8413
\]

2) Normal with $\mu = 2 \times 20 + 3 = 43$ and $\sigma^2 = 4 \times 25 = 100$

3)

\[
P \left( Z \leq \frac{x_0 - 20}{5} \right) = .05
\]
\[
\frac{x_0 - 20}{5} = -1.645
\]
\[
x_0 = 11.775
\]
5. (18 points) Let \( X_1 \) and \( X_2 \) be two random variables with joint probability function given by

\[
\begin{array}{c|c|c}
\text{Table Draw by hand}
\end{array}
\]

1) Find the marginal probability function of \( X_1 \).
2) Calculate \( E(X_1) \) and \( V(X_1) \).
3) Find \( E(X_2|X_1 = 1) \).

**Solution.** 1)

\[
p_1(-2) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}
\]

\[
p_1(1) = \frac{1}{6} + \frac{1}{2} = \frac{2}{3}
\]

2)

\[
E(X_1) = (-2)\frac{1}{3} + \frac{2}{3} = 0
\]

\[
V(X) = (-2)^2 \frac{1}{3} + \frac{2}{3} = 2
\]

3)

\[
p(-2|x_1 = 1) = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{4}
\]

\[
p(1|x_1 = 1) = \frac{\frac{1}{2}}{\frac{2}{3}} = \frac{3}{4}
\]

Thus

\[
E(X_2|X_1 = 1) = (-2)\frac{1}{4} + \frac{3}{4} = \frac{1}{4}
\]

6. (10 points) Suppose \( X_1 \) and \( X_2 \) have joint probability density function

\[
f(x_1, x_2) = 8x_1x_2, \quad 0 \leq x_1 \leq x_2 \leq 1.
\]

Calculate \( E(X_1X_2) \).

**Solution.**

\[
E(X_1X_2) = \int_0^1 \int_0^{x_2} x_1x_2 8x_1x_2 dx_1 dx_2
\]

\[
= \int_0^1 8x_2 \int_0^{x_2} x_1^2 dx_1 dx_2
\]

\[
= \int_0^1 8x_2^2 \frac{1}{3} x_2^3 dx_2
\]

\[
= \int_0^1 \frac{8}{3} x_2^5 dx_2 = \frac{4}{9}
\]