Chapter 6

10. (c) If \( x \in A \cap B \), then \( x \in A \) and \( x \in B \). Thus
\[
x \geq \inf A \text{ and } x \geq \inf B.
\]
Therefore
\[
x \geq \max(\inf A, \inf B).
\]
Hence
\[
\inf(A \cap B) \geq \max(\inf A, \inf B).
\]
(d) Take
\[
A = \{2\} \cup \left\{ 1 + \frac{1}{n} : n \in \mathbb{N} \right\} \text{ and } B = \{2\} \cup \left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\}.
\]
Then \( A \cap B = \{2\} \),
\[
\inf A = 1, \quad \inf B = 0 \text{ and } \inf(A \cap B) = 2 > \max(\inf A, \inf B).
\]

11. (b) Proof: As \(-E \subseteq \mathbb{Z}\) is nonempty and bounded above, \( \sup(-E) \in -E \).
Hence \(- \inf E \in -E\), which implies \( \inf E \in E \).

14. Proof: (b) As
\[
f'(x) = 2x + 2 > 0 \text{ for } x > -1,
\]
f decreases on \((-2, -1)\) and increases on \((-1, 2)\). Note
\[
f(-2) = 0, \quad f(-1) = -1 \text{ and } f(2) = 8.
\]
Hence
\[
\inf f(E) = \min f(E) = -1, \quad \sup f(E) = 8.
\]
(c) As
\[
f'(x) = \frac{-2}{x^3} > 0 \text{ for } x < 0,
\]
f increases on \([-1, 0)\) and decreases on \((0, 1]\). Note
\[
f(-1) = f(1) = 2, \quad f(0-) = f(0+) = \infty.
\]
Hence, there is no \( \sup f(E) \) and
\[
\inf f(E) = \min f(E) = 2.
\]
15. (a) Proof: As

\[ f(x) \leq \sup f(E) \text{ and } g(x) \leq \sup g(E), \]

we have

\[ f(x) + g(x) \leq \sup f(E) + \sup g(E). \]

Thus

\[ \sup(f + g)(E) \leq \sup f(E) + \sup g(E). \]

21. Proof: Since \( a - \frac{1}{n} < a + \frac{1}{n} \), there is \( r_n \in \mathbb{Q} \) such that

\[ a - \frac{1}{n} < r_n < a + \frac{1}{n}. \]

So

\[ |a - r_n| < \frac{1}{n}. \]
23. Proof: There is \( n \in \mathbb{N} \) such that

\[
\log_2 \frac{1}{b-a} < n, \text{ so } 1 < 2^n(b-a).
\]

Case 1: \( a \geq 0 \).

Let

\[
E = \{ k \in \mathbb{Z} : \frac{k}{2^n} \leq a \}.
\]

As \( 0 \in E \), \( E \) is not empty. As \( k \in E \) implies \( k \leq 2^n a \), \( E \) is bounded above. So \( k_0 = \sup E \) exists. By Exercise 11 in section 6.2, \( k_0 \in E \). Then

\[
\frac{k_0}{n} \leq a \text{ and } \frac{k_0 + 1}{n} > a.
\]

Let \( m = k_0 + 1 \). Then \( a < \frac{m}{2^n} \) and

\[
\frac{m}{2^n} = \frac{k_0}{2^n} + \frac{1}{2^n} \leq a + \frac{1}{2^n} \leq a + b - a = b.
\]

Case 2: \( a < 0 \).

As \( -a > 0 \), by Archimedean principle, there is \( k \in \mathbb{N} \) such that \( -a < k \).

So \( k + a > 0 \).

By case 1, there is \( r, n \in \mathbb{N} \) such that

\[
k + a < \frac{r}{2^n} < k + b.
\]

Let \( m = r - 2^n k \). Then

\[
a < \frac{m}{2^n} < b.
\]