44. b) If $\sqrt{3} = \frac{c}{d}$ (reduced) is rational, then $c^2 = 3d^2$ is divisible by 3. Let $c = 3k + r$ with $r = 0, 1, 2$. If $r = 1$, then $c^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$ is not divisible by 3. If $r = 2$, then $c^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$ is not divisible by 3. Hence $c = 3k$. Then $9k^2 = 3d^2$ and $d^2 = 3k^2$. There is $l \in \mathbb{N}$ such that $d = 3l$. So $c$, $d$ have common factor 3 which contradicts from that $\frac{c}{d}$ is reduced. Hence $\sqrt{3}$ is irrational.

47. If $a^{\frac{1}{n}} \geq b^{\frac{1}{n}}$, then $(a^{\frac{1}{n}})^n \geq (b^{\frac{1}{n}})^n$. Thus $a \geq b$. A contradiction!

50. As 

$$(a + b)^2 = a^2 + b^2 + 2ab \geq a^2 + b^2,$$

we have 

$$\left(\left((a + b)^2\right)^{\frac{1}{2}} \geq (a^2 + b^2)^{\frac{1}{2}}\right).$$

Thus 

$$\sqrt{a^2 + b^2} \leq a + b.$$

51. a) 

$$G(a,b) = \sqrt{ab} \geq \sqrt{aa} = a,$$

$$A(a,b) = \frac{a + b}{2} \leq \frac{b + b}{2} = b.$$ 

As 

$$A(a,b)^2 - G(a,b)^2 = \frac{(a + b)^2}{4} - ab = \frac{a^2 + 2ab + b^2 - 4ab}{4} = \frac{(a - b)^2}{4} \geq 0,$$

we have 

$$A(a,b) \geq G(a,b).$$

b) $A(a,b) = G(a,b)$ if and only if $\frac{(a-b)^2}{4} = 0$. Namely, $a = b$. 

1
56. a) $f$ is nowhere increasing.

b) Domain: $x \neq \pm 1$. Set
\[ f'(x) = \frac{-4x}{(x^2 - 1)^2} > 0, \]
we get $x < 0$. Thus $f$ is increasing on $(-\infty, -1)$ and $(-1, 0)$.

c) As
\[ f'(x) = 3x^2 \cos(x^3) > 0 \]
at $\sqrt[3]{2k\pi - \frac{\pi}{2}} < x < \sqrt[3]{2k\pi + \frac{\pi}{2}}$, $k = 0, \pm 1, \pm 2, \ldots$, $f$ is increasing on these intervals.

60. a) If $a < b$, then $f(a) < f(b)$ and $g(a) < g(b)$. So,
\[ f(a) + g(a) < f(b) + g(b). \]
Thus $f + g$ is strictly increasing.

It is inconclusive for $f g$ and $c f$. For example, if $f(x) = e^x$ and $g(x) = c = -1$, then $f g = c f = -e^x$ is decreasing; while for $g(x) = c = 1$, $f g = c f = e^x$ is increasing.

b) As $g$ is strictly decreasing, $-g$ is strictly increasing. So, $f - g = f + (-g)$ is strictly increasing.

If $a < b$, then $0 < f(a) < f(b)$ and $g(a) > g(b) > 0$. Then
\[ \frac{1}{g(a)} < \frac{1}{g(b)}. \]
Thus
\[ \frac{f(a)}{g(a)} < \frac{f(b)}{g(b)}. \]
$f / g$ is strictly increasing.

65. c) $P(x) = x^4 + 5x^3 - 2x^2 + x - 6$.

If $a = \frac{c}{d}$ is a rational root, then $c$ divides 6 and $d$ divides 1. So $a = \pm 1, \pm 2, \pm 3, \pm 6$.

\[ P(1) = -1, \quad P(-1) = -13, \quad P(2) = 44, \]
\[ P(-2) = -40, \quad P(3) = 195, \quad P(-3) = -81, \]
\[ P(6) = 2304, \quad P(-6) = 132. \]

There is no rational root.

d) \[ P(x) = 6x^3 - 23x^2 + 16x - 3. \]

If \( a = \frac{c}{d} \) is a rational root, then \( c \) divides 3 and \( d \) divides 6. So \( a = \pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{1}{3}, \pm \frac{1}{6} \).

\[ P(1) = -4, \quad P(-1) = -48, \quad P(3) = 0. \]

There is a factor \( x - 3 \). Use division, we get

\[ P(x) = (x - 3)(6x^2 - 5x + 1). \]

The other two solutions are:

\[ x = \frac{5 \pm \sqrt{25 - 24}}{12}. \]

The solutions are: \( x = 3, \frac{1}{2}, \frac{1}{3} \).

66 b) Let \( a = \sqrt[5]{13} \). Then \( a^5 = 13 \) so \( a \) is a root of \( P(x) = x^5 - 13 \). If \( a = \frac{c}{d} \) is a rational root, then \( c \) divides 13 and \( d \) divides 1. So \( a = \pm 1, \pm 13 \).

\[ P(1) = -12, \quad P(-1) = -14, \quad P(13) = 371280, \quad P(-13) = -371306. \]

There is no rational root so \( a \) is irrational.