Homework 10

12 b) Proof: As $a < b$ and $c > 0$, $ac < bc$.
As $c < d$ and $b > 0$, $bc < bd$. By transitivity, $ac < bd$. ■

14. Proof: Suppose that $0 > 1$. Then $0 \cdot 1 < 1 \cdot 1$. Thus $0 < 1$. This contradiction implies $0 < 1$.
As $0 < 1$, $0 - 1 < 1 - 1$. So $-1 < 0$. ■

22. Solution: As
\[
\frac{x - 2}{(x - 2)(x + 2)} > \frac{1}{2},
\]
x ≠ 2 and x ≠ −2. Further
\[
\frac{1}{x + 2} > \frac{1}{2},
\]
If $x > -2$, then $2 > x + 2$ and hence, $x < 0$. The solution in this case is $-2 < x < 0$.
If $x < -2$, then $2 < x + 2$ and hence, $x > 0$. No solution is this case.
Therefore, the solution is $-2 < x < 0$. ■

25. Proof: If $a = 0$ or $b = 0$, then $ab = 0$. Thus $a ≠ 0$ and $b ≠ 0$.
If $a > 0$, then $q(a) > 0$. Otherwise, $q(a) < 0$ and hence, $aq(a) < 0q(a)$ which implies $1 < 0$ contradicts from Problem 14. As $ab > 0$, $q(a)ab > q(a)0$ and hence $b > 0$.
If $a < 0$, then $b < 0$. Otherwise, $b > 0$. By similar argument as in last paragraph, we have $a > 0$. This contradiction implies $b < 0$. ■

Another Proof: Suppose not, then $a ≥ 0 ≥ b$ or $b ≥ a ≥ b$. By symmetry, we only need to consider the first case. Since $b ≤ 0$ and $a ≥ 0$, we have
\[
ab ≤ a0 = 0.
\]
This contradict from $ab > 0$. ■

29. c) If $x, y ≥ 0$, then $x + y = 1$.
If $x < 0$, $y ≥ 0$, then $-x + y = 1$.
If $x < 0$, $y < 0$, then $-x - y = 1$.
If $x ≥ 0$, $y < 0$, then $x - y = 1$.
32. If \( a \geq 0, \ b \geq 0 \), then \( ab \geq 0 \) and

\[ |ab| = ab = |a||b|. \]

If \( a \geq 0, \ b < 0 \), then \( ab < 0 \) and

\[ |ab| = -ab = a(-b) = |a||b|. \]

If \( a < 0, \ b \geq 0 \), then \( ab < 0 \) and

\[ |ab| = -ab = (-a)b = |a||b|. \]

If \( a < 0, \ b < 0 \), then \( ab \geq 0 \) and

\[ |ab| = ab = (-a)(-b) = |a||b|. \]

34: “If” If \( a = 0 \), then \( |a| = 0 < \epsilon \) for all \( \epsilon > 0 \).

“Only if” If \( a \neq 0 \), then \( |a| > 0 \). Let \( \epsilon = \frac{|a|}{2} \). Then \( \epsilon > 0 \) and \( |a| > \epsilon \).

This contradicts from \( |a| < \epsilon \) for all \( \epsilon > 0 \). Thus \( a = 0 \).

36.

\[ |x^2 + x - 2| = |(x + 2)(x - 1)| \]
\[ \leq (|x| + 2)|x - 1| \]
\[ \leq 4|x - 1|. \]