1. (a) Find the Fourier sine series, the Fourier cosine series and the full Fourier series expansion of \( e^x \) on \((0, 2)\) or \((-2, 2)\) as appropriate. (Note that once you’ve done the work for finding the sine and cosine series coefficients, you need only divide by 2 and change the limits of integration in the integrals used to find the coeffs for the full series - this will save you alot of work!)

(b) Use MATLAB to plot the approximation by each type of series (for example, using the full series we have \( f(x) \approx \frac{1}{2}A_0 + \sum_{n=1}^{N}(A_n \cos(n\pi x/l) + B_n \sin(n\pi x/l)) \) for \( N = 3, 5, 10, 100 \), each one plotted on the same axes along with a plot of the actual function \( f(x) = e^x \) (you should have one plot for each type of Fourier series). All of these plots should only be over the interval \([-2, 2]\) - and make sure you label each curve. [Let me know if you need some guidance on these MATLAB parts]

**Solution:**
For the sine series, we want to write
\[
e^x = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right)
\]
and we need to find the coefficients \( A_n \) that make this equality true.

\[
A_n = \int_0^2 e^x \sin\left(\frac{n\pi x}{2}\right) dx
\]

\[
= -\frac{2}{n\pi} e^x \cos\left(\frac{n\pi x}{2}\right)|_0^2 + \frac{2}{n\pi} \int_0^2 e^x \cos\left(\frac{n\pi x}{2}\right) dx
\]

\[
= -\frac{2}{n\pi} (e^2 (-1)^n - 1) + \frac{4}{n^2\pi^2} e^x \sin\left(\frac{n\pi x}{2}\right)|_0^2 - \frac{4}{n^2\pi^2} \int_0^2 e^x \sin\left(\frac{n\pi x}{2}\right) dx
\]

\[
= -\frac{2}{n\pi} (e^2 (-1)^n - 1) - \frac{4}{n^2\pi^2} \int_0^2 e^x \sin\left(\frac{n\pi x}{2}\right) dx
\]

We can add the final integral to both sides to get that
\[
(1 + \frac{4}{n^2\pi^2}) \int_0^2 e^x \sin\left(\frac{n\pi x}{2}\right) dx = -\frac{2}{n\pi} (e^2 (-1)^n - 1)
\]
or
\[
A_n = \int_0^2 e^x \sin\left(\frac{n\pi x}{2}\right) dx = -\frac{2n\pi}{n^2\pi^2 + 4} (e^2 (-1)^n - 1)
\]
Similarly, we can find the coeffs for the cosine series by:

\[ B_n = \int_0^2 e^x \cos\left(\frac{n\pi x}{2}\right) \, dx \]
\[ = \frac{2}{n\pi} e^x \sin\left(\frac{n\pi x}{2}\right) \bigg|_0^2 - \frac{2}{n\pi} \int_0^2 e^x \sin\left(\frac{n\pi x}{2}\right) \, dx \]
\[ = \frac{4}{n^2\pi^2} e^x \cos\left(\frac{n\pi x}{2}\right) \bigg|_0^2 - \frac{4}{n^2\pi^2} \int_0^2 e^x \cos\left(\frac{n\pi x}{2}\right) \, dx \]
\[ = \frac{4}{n^2\pi^2} \left(e^2(-1)^n - 1\right) - \frac{4}{n^2\pi^2} \int_0^2 e^x \cos\left(\frac{n\pi x}{2}\right) \, dx \]

We can add the final integral to both sides to get that

\[ (1 + \frac{4}{n^2\pi^2}) \int_0^2 e^x \sin\left(\frac{n\pi x}{2}\right) \, dx = \frac{4}{n^2\pi^2} \left(e^2(-1)^n - 1\right) \]

or

\[ B_n = \int_0^2 e^x \sin\left(\frac{n\pi x}{2}\right) \, dx = \frac{4}{n^2\pi^2 + 4}(e^2(-1)^n - 1) \]

and \( B_0 = \int_0^2 e^x \, dx = e^2 - 1. \)
If we use the same integration by parts, but change our limits of integration to \([-2, 2]\) in order to find the full Fourier series for \(e^x\), we get

\[
A_n = \frac{1}{2} \int_{-2}^{2} e^x \sin \left(\frac{n \pi x}{2}\right) \, dx
= -\frac{(-1)^n}{n \pi} \left(e^2 - e^{-2}\right) - \frac{2}{n^2 \pi^2} \int_{0}^{2} e^x \sin \left(\frac{n \pi x}{2}\right) \, dx
\]
or \(A_n = \frac{(-1)^n n \pi}{4 + n^2 \pi^2} (e^2 - e^{-2})\).  

Similarly,

\[
B_n = \frac{1}{2} \int_{-2}^{2} e^x \cos \left(\frac{n \pi x}{2}\right) \, dx
= \frac{2(-1)^n}{n^2 \pi^2} (e^2 - e^{-2}) - \frac{2}{n^2 \pi^2} \int_{-2}^{2} e^x \cos \left(\frac{n \pi x}{2}\right) \, dx
\]

so that \(B_n = \frac{2(-1)^n}{4 + n^2 \pi^2} (e^2 - e^{-2})\). We also have \(B_0 = \frac{1}{2} \int_{-2}^{2} e^x \, dx = \frac{e^2 - e^{-2}}{2}\).
(c) Now plot (one for each FS type) just the approximate Fourier series for $x \in [-10, 10]$ with $N = 10$. What do you notice? Explain the differences in what you see.

Solution
In the case of the sine series, we see the result we had on $[0, 2]$ mirrored over the line $x = y$ to $[-2, 0]$ (so that it is odd) and then the region from $[-2, 2]$ is repeated periodically over the whole line. This is because a sine series is always an odd function and of period $2l = 4$. 
In the case of the cosine series, we see the result we had on $[0, 2]$ mirrored over the $y$-axis to $[-2, 0]$ (so that it is even) and then the region from $[-2, 2]$ is repeated periodically over the whole line. This is because a cosine series is always an even function and of period $2l = 4$. 
Fourier Cosine Series for $e^x$ defined over $[-2,2]$ plotted for $x$ in $[-10,10]$

In the case of the full series, we see the plot for the full FS obtained for $(-2,2)$ repeated periodically of period 4 across the real line. This is because the full fourier series is always a periodic function of period $2l$, and the full fourier series was created to match $e^x$ on the fundamental period interval $(-2,2)$. 
2. Show that IF $U(x)$ is a (steady-state) solution to $U_{xx} = 0$ on $(0, l)$ with

$$
U(0) = g \\
U(l) = h
$$

for some fixed constants $g, h$, and IF $\tilde{u}$ is a solution to $\tilde{u}_{xx} = \tilde{u}_t$ on $(0, l)$ with

$$
\tilde{u}(0, t) = 0 \\
\tilde{u}(l, t) = 0
$$

where $\tilde{u}(x, 0) = f(x) - U(x)$, THEN $u(x, t) = \tilde{u}(x, t) + U(x)$ solves $u_{xx} = u_t$ where

$$
\begin{align*}
    u(0, t) &= g \\
    u(l, t) &= h
\end{align*}
$$

and $u(x, 0) = f(x)$.

[**NOTE: The point of this problem is that it allows us to solve BVP’s with nonhomogeneous boundary conditions by building a solution from the homogeneous b.c. problem and the corresponding steady-state problem... Notice that the seperation of variables technique breaks down if we have inhomogeneous b.c.’s]

**Solution:**

Letting $u = \tilde{u} + U$, we have $u_t = \tilde{u}_t + U_t = \tilde{u}_t + 0$ since $U$ is independent of $t$. Also, we have $u_{xx} = \tilde{u}_{xx} + U_{xx} = \tilde{u}_{xx} + 0$, since $U_{xx} = 0$. By the PDE for $\tilde{u}$, we then have $u_t = u_{xx}$. 
Finally, \( u(0, t) = \tilde{u}(0, t) + U(0) = 0 + g = g \) and \( u(l, t) = \tilde{u}(l, t) + U(l) = 0 + h = h \). The initial condition is obtained by \( u(x, 0) = \tilde{u}(x, 0) + U(x) = f(x) - U(x) + U(x) = f(x) \).

3. Solve problem 8 from section 5.1 of Strauss using exercise one above.

The system we want to solve is: \( u_t = u_{xx} \) on \([0, 1]\), with conditions \( u(0, t) = 0, u(1, t) = 1, u(x, 0) = 5x/2 \) for \( x \in (0, 2/3) \) and \( u(x, 0) = 3 - 2x \) for \( x \in (2/3, 1) \). The steady state solution to this problem satisfies \( U_{xx} = 0 \) and \( U(0) = 0, U(1) = 1 \), so \( U(x) = x \). We need to then solve the homogeneous dirichlet bc heat equation \( \tilde{u}_t = \tilde{u}_{xx} \) with \( \tilde{u}(0, t) = 0 \) and \( \tilde{u}(1, t) = 0 \), and \( \tilde{u}(x, 0) = f(x) - x \). We know the general solution to this BVP is \( u(x, t) = \sum_{n=1}^{\infty} A_n e^{-n^2 \pi^2 t} \sin(n \pi x) \), so applying the initial condition we have

\[
\begin{align*}
\sum_{n=1}^{\infty} A_n \sin(n \pi x) = f(x) - x.
\end{align*}
\]

We can find the coefficients \( A_n \) by recognizing this has the form of a Fourier sine series, and so

\[
A_n = 2 \int_0^1 (f(x) - x) \sin(n \pi x) \, dx = 2 \int_0^{2/3} \frac{3x}{2} \sin(n \pi x) \, dx + 2 \int_{2/3}^1 (3 - 3x) \sin(n \pi x) \, dx.
\]

Integrating we have

\[
A_n = 3 \left( \frac{-x}{n \pi} \cos(n \pi x) \bigg|_0^{2/3} + \frac{1}{n^2 \pi^2} \sin(n \pi x) \bigg|_0^{2/3} \right)
+ 6 \left( \frac{-1}{n \pi} \cos(n \pi x) \bigg|_0^{1/2} + \frac{x}{n \pi} \cos(n \pi x) \bigg|_0^{1/2} - \frac{1}{n^2 \pi^2} \cos(n \pi x) \bigg|_0^{1/2} \right)
+ \left( \frac{-6(1)^n}{n \pi} + \frac{6}{n \pi} \cos(2n \pi / 3) + \frac{6(1)^n}{n \pi} - \frac{4}{n \pi} \cos(2n \pi / 3) - \frac{6(1)^n}{n^2 \pi^2} + \frac{6}{n^2 \pi^2} \cos(2n \pi / 3) \right)
= \frac{6}{n^2 \pi^2} \cos(2n \pi / 3) - \frac{6(1)^n}{n^2 \pi^2} + \frac{3}{n^2 \pi^2} \sin(2n \pi / 3)
\]

4. A string (with density \( \rho = 1 \) and tension \( T = 4 \)) with fixed ends at \( x = 0 \) and \( x = 10 \) is hit by a hammer so that \( u(x, 0) = 0 \) and

\[
\frac{\partial u}{\partial t}(x, 0) = \begin{cases} V & \text{if } x \in [-\delta + 5, \delta + 5] \\ 0 & \text{otherwise}. \end{cases}
\]

Find the height of the string \( u(x, t) \) for all \( x \in (0, 10) \) and all \( t > 0 \). (Your answer WILL be a bit messy...)

**Solution**

Our solution to the BVP is \( u(x, t) = \sum_{n=1}^{\infty} (A_n \sin(\frac{n \pi t}{5}) + B_n \cos(\frac{n \pi t}{5})) \sin(\frac{n \pi x}{10}) \). Applying the first initial condition we have

\[
u(x, 0) = \sum_{n=1}^{\infty} B_n \sin(\frac{n \pi x}{10}) = 0
\]
so that $B_n = 0$ for every $n$. Applying the second initial condition, we have

$$\frac{\partial u}{\partial t}(x, 0) = \sum_{n=1}^{\infty} \frac{n\pi A_n}{5} \sin\left(\frac{n\pi x}{10}\right) \begin{cases} V & \text{if } x \in [-\delta + 5, \delta + 5] \\ 0 & \text{otherwise} \end{cases}$$

We can now use the fact that we have a Fourier sine series here to find the coefficients $A_n$. We will need

$$\frac{n\pi A_n}{5} = \frac{2}{10} \int_{5-\delta}^{5+\delta} V \sin\left(\frac{n\pi x}{10}\right) dx = -\frac{2V}{n\pi} \cos\left(\frac{n\pi x}{10}\right)_{5-\delta}^{5+\delta}$$

Thus,

$$A_n = -\frac{10V}{n^2\pi^2} \left( \cos\left(\frac{n\pi}{2} + \frac{n\pi \delta}{10}\right) - \cos\left(\frac{n\pi}{2} - \frac{n\pi \delta}{10}\right) \right)$$

5. Problem 15 section 5.2 of Strauss.

**Solution**

Since $\sin(x)$ is an even function, the coefficients for the sine terms in the full Fourier series will vanish ($=0$) and we will have a pure cosine series.

6. Using parts of our discussion in class, solve the fourth order equation $u_{xxxx} = u_t$ if $u(0, t) = 0$, $u(3, t) = 0$, $u_{xx}(0, t) = 0$, and $u_{xx}(3, t) = 0$.

**Solution**

Using separation of variables, we have $X^{(4)} = T'' = \lambda$. We know by our work in class that since we have the above homogeneous boundary conditions, we will have eigenvalues $\lambda \geq 0$ only. So, we will check the cases $\lambda = 0$ and $\lambda > 0$. If $\lambda = 0$, we have $X(x) = ax^3 + bx^2 + cx + d$ and applying our boundary conditions we get: $d = 0$, $27a + 9b + 3c = 0$, $2b = 0$ and $18a = 0$. Thus $a = b = c = d = 0$ and we get only the trivial solution.

Now we check $\lambda = \beta^4 = 0$. Then the characteristic equation for $X^{(4)} - \beta^4 X = 0$ is $r^4 - \beta^4 = 0$. We can factor this as $(r^2 - \beta^2)^2 = 0$, so that the roots are $r = \pm \beta$, $\pm i\beta$ and $X(x) = c_1 e^\beta x + c_2 e^{-\beta x} + c_3 \cos(\beta x) + c_4 \sin(\beta x)$. Applying the first three boundary conditions we get:

$$c_1 + c_2 + c_3 = 0$$
$$c_1 e^{3\beta} + c_2 e^{-3\beta} + c_3 \cos(3\beta) + c_4 \sin(3\beta) = 0$$
$$\beta^2 (c_1 + c_2 - c_3) = 0$$

This tells us that $c_3 = 0$ and so $c_2 = -c_1$. Subbing that into the second equation gives $c_1 (e^{3\beta} - e^{-3\beta}) + c_4 \sin(3\beta) = 0$. Now applying our final condition gives $\beta^2 (c_1 e^{3\beta} - c_1 e^{-3\beta} - c_4 \sin(3\beta)) = 0$. This leads us to conclude that $c_1 = 0$ and so $c_4 \sin(3\beta) = 0$. Our only hope for a nontrivial solution is to have $3\beta = n\pi$ or $\beta = \frac{n\pi}{3}$. Thus we get infinitely many solutions for $X$, one for each $n$, and

$$X_n(x) = C_n \sin\left(\frac{n\pi x}{3}\right).$$

Now we can solve for the corresponding functions $T_n$ with $\lambda_n = \beta^4 = \left(\frac{n\pi}{3}\right)^4$. $T'_n = \frac{n^4 \pi^4}{81} T_n$, so that $T_n = D_n e^{\frac{n^4 \pi^4}{81} t}$. Our final solution is then

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{\frac{n^4 \pi^4}{81} t} \sin\left(\frac{n\pi x}{3}\right).$$
Aside: Because our BC’s meet the symmetry condition for operator $L(u) = u_{xxxx}$ which we discussed in class, we know that if we had an initial condition, at this point we can use the Fourier series method to uniquely determine the coefficients $A_n$. 