1. Solve the heat equation (i.e. - the diffusion equation) $4u_{xx} = u_t$ on a rod of length 2 if $u(x,0) = sin\left(\frac{\pi x}{2}\right)$ and $u(0,t) = 0 = u(2,t)$. 

**Solution:**

We are solving the heat equation on a finite interval $(0,2)$, with dirichlet boundary conditions, so we can use the general solution to this boundary value problem that we derived in class via separation of variables:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2 \pi^2 t} \sin(n \pi x/2).$$

In order to finish, we need to determine the values of the $A_n$'s. Since $u(x,0) = \sum_{n=1}^{\infty} A_n \sin(n \pi x/2) = \sin(\pi x/2)$ we can take $A_1 = 1$ and $A_n = 0$ for all $n \neq 1$. Thus

$$u(x,t) = e^{-\pi^2 t} \sin(\pi x/2).$$

2. Solve the wave equation $3u_{xx} = u_{tt}$ for a clamped string of length $l = 1$ (so $u(0,t) = 0 = u(1,t)$) such that $u(x,0) = 2 \sin(\pi x) \cos(\pi x)$ and $u_t(x,0) = 0$. [hint: use a double angle identity from trig]

Again, we are solving the wave equation on a finite length interval $(0,1)$ with dirichlet boundary conditions, so since we have already solved this general boundary value problem, we can use the solution we obtained via separation of variables:

$$u(x,t) = \sum_{n=1}^{\infty} \left( a_n \cos(n \pi \sqrt{3} t) + b_n \sin(n \pi \sqrt{3} t) \right) \sin(n \pi x).$$

We need to determine the values of the $a_n$'s and $b_n$'s in order to have solved our problem completely. Since $u(x,0) = \sum_{n=1}^{\infty} a_n \sin(n \pi x) = 2 \sin(\pi x) \cos(\pi x) = \sin(2 \pi x)$ we see that we can take $a_2 = 1$ and $a_n = 0$ for $n \neq 2$. This gives

$$u(x,t) = \cos(2 \pi \sqrt{3} t) \sin(2 \pi x) + \sum_{n=1}^{\infty} b_n \sin(n \pi \sqrt{3} t) \sin(n \pi x).$$

It follows that

$$u_t(x,t) = -2 \pi \sqrt{3} \sin(2 \pi \sqrt{3} t) \sin(2 \pi x) + \sum_{n=1}^{\infty} n \pi \sqrt{3} b_n \cos(n \pi \sqrt{3} t) \sin(n \pi x),$$

and

$$u_t(x,0) = \sum_{n=1}^{\infty} n \pi \sqrt{3} b_n \sin(n \pi x) = 0.$$

This tells us that we can take $b_n = 0$ for all $n$. Finally, we have

$$u(x,t) = \cos(2 \pi \sqrt{3} t) \sin(2 \pi x).$$
3. Strauss Exercise 4, pg 87 (solve by separation of variables, in the same way that we did in class)

Letting \( u(x, t) = X(x)T(t) \) and subbing in, we have

\[
\frac{T'' + rT'}{c^2T} = \frac{X''}{X} = \lambda
\]

so that

\[
X'' = \lambda X
\]

\[
T'' + rT' = \lambda c^2 T .
\]

We again have Dirichlet boundary conditions, so the solution for \( X \) is only nontrivial if \( \lambda = -\beta^2 < 0 \), and we get \( X(x) = c_1 \cos(\beta x) + c_2 \sin(\beta x) \). Applying the boundary conditions, we have \( X(0) = c_1 = 0 \) and \( X(l) = c_2 \sin(\beta l) = 0 \). In order to get a nontrivial solution, we then need that \( \beta l = n\pi \) for some \( n \in \mathbb{Z} \), or \( \beta = n\pi/l \). This tells us that we have a solution \( X_n \) for each integer \( n \), and

\[
X_n(x) = c_n \sin(n\pi x/l) .
\]

Now we can solve for the corresponding functions \( T_n \). We can try \( T_n(t) = e^{kt} \). This gives

\[
k^2 + rk - c^2 \lambda = 0
\]

which has as it’s solutions

\[
k = \frac{-r \pm \sqrt{r^2 + 4c^2\lambda}}{2} = \frac{-r \pm \sqrt{r^2 - 4c^2n^2\pi^2/l^2}}{2} .
\]

The types of solutions we get then depend on whether or not \( k \) is real or complex, which is determined by the sign of \( r^2 - 4c^2n^2\pi^2/l^2 \). Since we are given that \( 0 < r < 2\pi c/l \) that implies that \( r^2 < 4\pi^2 c^2/l^2 \) and since \( n \geq 1 \), we get \( r^2 < 4\pi^2 c^2n^2/l^2 \), or \( r^2 - 4c^2n^2\pi^2/l^2 < 0 \). Hence \( k \) is complex and the solutions are

\[
T_n(t) = e^{-rt/2} \left(a_n \cos\left(\frac{\sqrt{4c^2n^2\pi^2/l^2 - r^2}}{2} t\right) + b_n \sin\left(\frac{\sqrt{4c^2n^2\pi^2/l^2 - r^2}}{2} t\right)\right) .
\]

This gives us, by the linearity of the PDE and the superposition principle, that the general solution is

\[
u(x, t) = \sum_{n=1}^{\infty} e^{-rt/2} \left(a_n \cos\left(\frac{\sqrt{4c^2n^2\pi^2/l^2 - r^2}}{2} t\right) + b_n \sin\left(\frac{\sqrt{4c^2n^2\pi^2/l^2 - r^2}}{2} t\right)\right) \sin(n\pi x/l) .
\]

Now to determine the coefficients, we note that

\[
u(x, 0) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/l) = \phi(x) ,
\]

and

\[
u_t(x, 0) = \sum_{n=1}^{\infty} \left(-\frac{r}{2} a_n + \frac{\sqrt{4c^2n^2\pi^2/l^2 - r^2}}{2} b_n\right) \sin(n\pi x/l) = \psi(x) .
\]

so that the \( a_n \)’s and \( b_n \)’s can be found by the method Fourier sine coefficients.

[NOTE: the neat thing about this problem is that you can directly see that this really does give you a damped wave - because of the factor \( e^{-rt/2} \) multiplying onto every term, as \( t \to \infty \), \( u(x, t) \to 0 \). This is quite different from the solution to the nondamped wave equation, where waves perpetuate indefinitely, with no decrease in amplitude.]
4. Straus, Exercise 6, pg 89.

We let \( u(x,t) = X(x)T(t) \) and substitute in the PDE \( tu_t = u_{xx} + 2u \). This yields \( tXT' = X''T + 2XT \), which can be separated into the two ODE's:

\[
\frac{tT'}{T} - 2 = \lambda
\]

and

\[
\frac{X''}{X} = \lambda.
\]

Since we have homogeneous Dirichlet Boundary conditions, and we are working with our usual ODE for \( X(x) \), we know that \( X_n(x) = C_n \sin(nx) \) for each \( n \) in the integers are all the possible solutions. Now solving the ODE for \( T \), we have

\[
tT' - (2 + \lambda)T = 0,
\]

or

\[
T' - \frac{2 + \lambda}{t} T = 0.
\]

Separating variables, we get:

\[
\frac{T'}{T} = \frac{(\lambda + 2)}{t}
\]

so integrating both sides yields

\[
\ln |T| = (\lambda + 2)\ln|t| + C
\]

or

\[
T = Ce^{(\lambda+2)\ln|t|} = Ct^{\lambda+2}
\]

Since there is a value of \( \lambda \) for each integer \( n \) by

\[
\lambda = -n^2
\]

, we have

\[
T_n(t) = D_n t^{-n^2+2}.
\]

Thus

\[
u(x,t) = \sum_{n=1}^{\infty} A_n t^{(-n^2+2)} \sin(nx)
\]

is the general solution to our BVP.

Now, applying the initial condition we see

\[
u(x,0) = 0
\]

*regardless* of our choices of the values for \( A_n \)’s! So ANY values of \( A_n \)’s work and we get infinitely many possible solutions to the IBVP. This problem is not well-posed.

5. Straus, exercise 1, page 92

We have \( kX''T = XT' \), so that

\[
\frac{X''}{X} = \frac{T'}{kT} = \lambda
\]
. We know that the general solution for $X$ is $X = c_1 e^{rt} + c_2 e^{-rt}$, where $r = \pm \sqrt{\lambda}$. We can first check the case where $\lambda > 0$, or $\lambda = \beta^2$. This yields $X(x) = c_1 e^{\beta x} + c_2 e^{-\beta x}$ and we can apply the boundary conditions $X(0) = 0$ and $X'(l) = 0$. We then have $c_1 = -c_2$ and $c_1 \beta (e^{\beta l} + e^{-\beta l}) = 0$. In order for the latter to be true, we need $c_1 = 0$ and so we have only the trivial solution $X(x) = 0$ for all $x$.

Now we can check the case for $\lambda = 0$. This gives $X(x) = cx + d$, and applying the boundary conditions we have $d = 0$ and $c = 0$, so that again we get only the trivial solution.

Finally we look at the case $\lambda < 0$ or $\lambda = -\beta^2$. This yields $X(x) = c_1 \cos(\beta x) + c_2 \sin(\beta x)$. Applying $X(0) = 0$ gives $c_1 = 0$. Applying next that $X'(l) = 0$ gives $c_2 \beta \cos(\beta l) = 0$ so that in order to obtain something nontrivial, we must take $\beta l = \frac{(2n-1)\pi}{2l}$, or $\beta = \frac{(2n-1)\pi}{2l}$. We then see we have an infinite family of solutions $X_n(x) = c_n \sin \left( \frac{(2n-1)\pi x}{2l} \right)$.

We can proceed to find the solutions $T_n(t)$ associated to each $X_n(x)$. For a fixed $n$, $\lambda = -\frac{(2n-1)^2 \pi^2}{4l^2}$, so the equation for $T_n$ is

$$T''_n = -\frac{k(2n-1)^2 \pi^2}{4l^2} T_n$$

and the solution is $T_n(t) = D_n e^{-\frac{k(2n-1)^2 \pi^2}{4l^2} t}$. So, for each $n$, we have a solution $u_n(x, t) = A_n e^{-\frac{k(2n-1)^2 \pi^2 x^2}{4l^2}} \sin \left( \frac{(2n-1)\pi x}{2l} \right)$ to the boundary value problem, and the general solution is then (by the superposition principle and the fact that our PDE is linear)

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-\frac{k(2n-1)^2 \pi^2 x^2}{4l^2}} \sin \left( \frac{(2n-1)\pi x}{2l} \right).$$

We are given no initial condition for this problem, so we are done!

6. Show that IF $U(x)$ is a (steady-state) solution to $U_{xx} = 0$ on $(0, l)$ with

$$U(0) = g$$
$$U(l) = h$$

for some fixed constants $g, h$, and IF $\tilde{u}$ is a solution to $\tilde{u}_{xx} = \tilde{u}_t$ on $(0, l)$ with

$$\tilde{u}(0, t) = 0$$
$$\tilde{u}(l, t) = 0$$

where $\tilde{u}(x, 0) = f(x) - U(x)$, THEN $u(x, t) = \tilde{u}(x, t) + U(x)$ solves $u_{xx} = u_t$ where

$$u(0, t) = g$$
$$u(l, t) = h$$

and $u(x, 0) = f(x)$.

[**NOTE: The point of this problem is that it allows us to solve BVP’s with nonhomogeneous boundary conditions by building a solution from the homogeneous b.c. problem and the corresponding steady-state problem... Notice that the separation of variables technique breaks down if we have inhomogeneous b.c.’s]
7. Solve problem 8 from section 5.1 of Strauss using exercise 6 above.

Solving the steady state system for \( U \), we get

\[
\int U_{xx} \, dx = \int 0 \, dx
\]

implies

\[
U_x = C_1
\]

and then integrating again, we get

\[
U(x) = C_1 x + C_2.
\]

Applying the boundary conditions \( U(0) = 0 \) and \( U(1) = 1 \) results in \( U(x) = x \).

Now, we need to find the solution to the corresponding homogeneous problem \( \tilde{u} \). Since it satisfies \( \tilde{u}_t = \tilde{u}_{xx} \) on \((0, 1)\) with \( \tilde{u} = \phi(x) - x \) and \( \tilde{u}(0, t) = 0 = \tilde{u}(1, t) \), we know that the solution can be found by separation of variables. Since we have Dirichlet boundary conditions and it’s the heat equation, we get

\[
\tilde{u}(x, t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) e^{-n^2 \pi^2 t}.
\]

Now solving for the \( A_n \)'s can be done by the standard means of finding Fourier sine series coefficients, since

\[
\tilde{u}(x, 0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) = \phi(x) - x.
\]

This means that

\[
A_n = \int_{0}^{1} (\phi(x) - x) \sin(n\pi x) \, dx
\]

Since

\[
\phi(x) = \begin{cases} 
\frac{5x}{2} & \text{for } 0 < x < 2/3 \\
3 - 2x & \text{for } 2/3 < x < 1
\end{cases}
\]

we get

\[
\phi(x) - x = \begin{cases} 
\frac{2x}{2} & \text{for } 0 < x < 2/3 \\
3 - 3x & \text{for } 2/3 < x < 1
\end{cases}
\]

so

\[
A_n = \int_{0}^{2/3} \frac{3x}{2} \sin(n\pi x) \, dx + \int_{2/3}^{1} (3 - 3x) \sin(n\pi x) \, dx.
\]

Now if we use integration by parts, we can find the generic integral

\[
\int_{a}^{b} x \sin(n\pi x) \, dx = -\frac{x \cos(n\pi x)}{n\pi} \bigg|_{a}^{b} + \int_{a}^{b} \frac{1}{n\pi} \cos(n\pi x) \, dx
\]

\[
= \frac{1}{n\pi} [-b \cos(n\pi b) + a \cos(n\pi a)] + \frac{1}{n^2 \pi^2} [\sin(n\pi b) - \sin(n\pi a)]
\]
which we can use to get $A_n$. Subbing in we get

$$A_n = \frac{3}{2} \left( - \frac{2}{3n\pi} \cos(2n\pi/3) + \frac{1}{n^2\pi^2} \sin(2\pi n/3) \right) - \frac{3}{n\pi} [\cos(n\pi) - \cos(2n\pi/3)]$$

$$- 3 \left( \frac{1}{n\pi} (-\cos(n\pi) + \frac{2}{3} \cos(2n\pi/3)) - \frac{1}{n^2\pi^2} \sin(2n\pi/3) \right).$$

This simplifies to

$$A_n = \frac{9}{2n^2\pi^2} \sin(2n\pi/3).$$

Plugging these coefficients into the expansion for $\tilde{u}$ defines $\tilde{u}$ completely. Finally we get $u(x, t) = \tilde{u} + x$ to be the solution to our inhomogeneous problem.

8. A string (with density $\rho = 1$ and tension $T = 4$) with fixed ends at $x = 0$ and $x = 10$ is hit by a hammer so that $u(x, 0) = 0$ and

$$\frac{\partial u}{\partial t}(x, 0) = \begin{cases} V & \text{if } x \in [-\delta + 5, \delta + 5] \\ 0 & \text{otherwise} \end{cases}.$$

Find the height of the string $u(x, t)$ for all $x \in (0, 10)$ and all $t > 0$. (Your answer WILL be a bit messy...)

**Solution:**

Again, we have the wave equation with dirichlet boundary conditions, so that the solution looks like

$$u(x, t) = \sum_{n=1}^{\infty} (A_n \sin(2n\pi t/10) + B_n \cos(2n\pi t/10)) \sin(n\pi x/10).$$

In order to satisfy our initial conditions, we note that

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/10) = 0,$$

which tells us that $B_n = 0$ for all $n$, and

$$u(x, t) = \sum_{n=1}^{\infty} A_n \sin(n\pi t/5) \sin(n\pi x/10).$$

Now for the initial velocity:

$$u_t(x, t) = \sum_{n=1}^{\infty} \frac{n\pi A_n}{5} \cos(n\pi t/5) \sin(n\pi x/10),$$

so

$$u_t(x, 0) = \sum_{n=1}^{\infty} \frac{n\pi A_n}{5} \sin(n\pi x/10).$$

Thus we are expanding our initial velocity in a fourier sine series, so the coefficients are

$$\frac{n\pi}{5} A_n = \frac{1}{5} \int_0^{10} u_t(x, 0) \sin(n\pi x/10) \, dx.$$
and using the definition of $u_t(x, 0)$ we get
\[
\frac{n\pi}{5} A_n = \frac{1}{5} \int_{5-\delta}^{5+\delta} V \sin(n\pi x/10) \, dx .
\]
Doing the computation gives
\[
A_n = \frac{10V}{n^2\pi^2} \left[ \cos((5 - \delta)n\pi/10) - \cos((5 + \delta)n\pi/10) \right] .
\]
To this point is fine, but we could also simplify further using the fact that
\[
\cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b) .
\]
This yields
\[
A_n = \frac{20V}{n^2\pi^2} \sin(5n\pi/10) \sin(\delta n\pi/10)
\]
which we can sub into
\[
u(x, t) = \sum_{n=1}^{\infty} A_n \sin(n\pi t/5) \sin(n\pi x/10)
\]
to get our final solution $u$.

9. Problems 5a and 6a from section 5.1 of Strauss, relying on the FS (sine) we found for $f(x) = x$ on $(0, l)$ in class (and in the book).

(5a) Since $x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2}{n\pi} \sin(n\pi x/l)$, we can integrate the series term-by-term to get
\[
\frac{x^2}{2} = \sum_{n=1}^{\infty} \frac{(-1)^n 2l^2}{n^2\pi^2} \cos(n\pi x/l) + C .
\]
So, it just remains to determine $C$. Note that this gives us a Fourier cosine series for $\frac{x^2}{2}$, so the $C$ should be the same as the $\frac{1}{l^2}A_0$ of the cosine series. Thus, since
\[
A_0 = \frac{2}{l^2} \int_0^l \frac{x^2}{2} \, dx = l^2 3
\]
we get $C = l^2 6$, and
\[
\frac{x^2}{2} = \frac{l^2}{6} + \sum_{n=1}^{\infty} \frac{(-1)^n 2l^2}{n^2\pi^2} \cos(n\pi x/l) .
\]

(6a) Now, we can do basically the same thing to get a Fourier series expansion for $x^3$. Integrating the series for $\frac{x^3}{2}$ term-by-term gives
\[
\frac{x^3}{6} = \frac{l^2}{6} x + \sum_{n=1}^{\infty} \frac{(-1)^n 2l^3}{n^3\pi^3} \sin(n\pi x/l) + C .
\]
Thus
\[
x^3 = l^2 x + \sum_{n=1}^{\infty} \frac{(-1)^n 12l^3}{n^3\pi^3} \sin(n\pi x/l) + C .
\]
Again, we still need to determine $C$, but subbing in zero to both sides shows us that $C = 0$. We aren’t quite done because the $l^2 x$ term makes the right hand side not quite a Fourier series. If we sub in the sine series for $x$, we get

$$x^3 = l^2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2l}{n\pi} \sin(n\pi x/l) + \sum_{n=1}^{\infty} \frac{(-1)^n 12l^3}{n^3\pi^3} \sin(n\pi x/l),$$

or

$$x^3 = \sum_{n=1}^{\infty} (-1)^n \left( \frac{-2l^3}{n\pi} + \frac{12l^3}{n^3\pi^3} \right) \sin(n\pi x/l).$$

10. Problem 15 section 5.2 of Strauss.

Since $|\sin(x)|$ is an even function, the sine coefficients for the full Fourier series over $(-\pi, \pi)$ will be zero. This is because determination of these coefficients are obtained by integrating

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |\sin(x)| \sin(nx) \, dx$$

and the fact that $\sin(nx)$ is an odd function and the product of an even and odd function is again odd, tells us that this integral will be zero, regardless of the value of $n$. 