1. Suppose that we have a uniform thin tube (approximable by one space dimension) of liquid with some particles which are suspended in the liquid. If the liquid is flowing through the pipe at a constant rate $c$ (m/s) and if we also take into account that the particles diffuse within the solution, derive the PDE for the concentration of the particles $u(x,t)$.

2. Suppose now that we have a still fluid in a tube, and again have particles suspended in that liquid. The particles move by diffusion AND sediment out of the solution at a fixed percentage rate $v$ (in units 1/s - $v$ is the fraction of particles that fall out of solution per second). Derive the PDE modeling the concentration of the particles $u(x,t)$.

3. Suppose that a uniform rod (approximable as one-dimensional) has a uniform heat source, so that the basic equation describing heat flow within the rod is

$$u_t = \alpha^2 u_{xx} + 1$$

for $0 \leq x \leq 1$. Suppose we fix the boundaries’ temperatures so that at $x = 0$ the rod is held at temperature 0 and at $x = 1$ the rod is held at temperature 1.

(a) Formulate the boundary conditions for the given problem.
(b) Write the boundary value problem (meaning the PDE and the boundary conditions) that describes the steady-state temperature of the rod.
(c) Use ODE techniques to solve the steady-state problem, if possible.

4. (a) What is a physical interpretation of the initial-boundary-value problem:

$$u_t = \alpha^2 u_{xx} \quad \text{for } 0 \leq x \leq 1, \quad 0 < t < \infty$$

$$u(0, t) = 0$$

$$u_x(1, t) = 1 \quad \text{for } 0 < t < \infty$$

$$u(x, 0) = \sin(\pi x) \quad \text{for } 0 \leq x \leq 1$$

(b) Can the solution come to a steady state? [hint: try to find steady-state solutions]
(c) Answer (a) and (b) again, but with the boundary conditions

$$u_x(0, t) = 0$$

$$u_x(1, t) = 0 \quad \text{for } 0 < t < \infty$$

5. (a) What is a physical interpretation of the initial-boundary-value problem:

$$u_{tt} = c^2 u_{xx} \quad \text{for } 0 \leq x \leq 1, \quad 0 < t < \infty$$

$$u(0, t) = 0$$

$$u(1, t) = \sin(t) \quad \text{for } 0 < t < \infty$$

$$u(x, 0) = 0$$

$$u_t(x, 0) = 0 \quad \text{for } 0 \leq x \leq 1$$

(b) Can the solution come to a steady state?
6. Consider the Neumann problem

\[ \Delta u = f(x, y, z) \quad \text{in} \ D \]
\[ \frac{\partial u}{\partial n} = 0 \quad \text{on} \ \partial D \]

(a) Show that if you have one solution \( u_1 \) to this boundary value problem, you can add any constant \( C \) and get a new solution \( u_C = u_1 + C \). [Thus we never have unique solutions for this problem!] What does this tell us about the actual physical system, if it is modeling steady state heat flow? Can you sort out why it is physically correct that we should *not* have unique solutions for this boundary value problem?

(b) Use the divergence theorem

\[ \int \int \int_D \nabla \cdot \vec{F} \, dV = \int \int \int_{\partial D} \vec{F} \cdot \vec{n} \, dS \]

and the PDE to show that we must have

\[ \int \int_D \int f(x, y, z) \, dx \, dy \, dz = 0 \]

in order for the boundary value problem to have a solution at all. Interpret what this says physically assuming the PDE represents steady state heat flow.