1. (a) [5 points] Is \( u_x + xu_y = 0 \) linear? Either way, prove that your claim is true.

Yes since \( f(u) = u_x + xu_y \) implies

\[
f(u + v) = (u + v)_x + x(u + v)_y = u_x + v_x + xu_y + xv_y = (u_x + xu_y) + (v_x + xv_y) = f(u) + f(v)
\]

and

\[
f(cv) = (cv)_x + x(cv)_y = cu_x + cxu_y = c(u_x + xu_y) = c f(u)
\]

(b) [5 points] Is \( u_x + u_y + 1 = 0 \) homogeneous? What is its order? Why?

No, it is inhomogeneous, since writing it in standard form gives \( u_x + u_y = -1 \), so that the RHS is not zero.

Its order is 1 since the highest order derivative that appears is a first order derivative.

(c) [5 points] Show that \( u(x,y) = 9x^2 + 6xy + y^2 \) is a solution to the PDE \( u_x - 3u_y = 0 \).

\[
egin{align*}
u_x &= 18x + 6y, \\
u_y &= 6x + 2y
\end{align*}
\]

\[
u_x - 3u_y = (18x + 6y) - 3(6x + 2y) = 18x + 6y - 18x - 6y = 0
\]

and so \( u \) is a soln to the PDE.

(d) [5 points] Show that if \( u_1, u_2, \ldots, u_n \) are all solutions to \( L(u) = 0 \), then \( v = c_1 u_1 + \cdots + c_n u_n \) is also a solution for any choice of constants \( c_i \), for all \( i \).

\( u_1, u_2, \ldots, u_n \) are all solutions to \( L(u) = 0 \), meaning \( L(u_i) = 0 \) for all \( i = 1, \ldots, n \).

If \( v = c_1 u_1 + \cdots + c_n u_n \), for some constants \( c_i \), then since \( L \) is linear:

\[
L(v) = L(c_1 u_1 + \cdots + c_n u_n) \\
= c_1 L(u_1) + c_2 L(u_2) + \cdots + c_n L(u_n) \\
= c_1 \cdot 0 + c_2 \cdot 0 + \cdots + c_n \cdot 0 \\
= 0
\]

thus \( L(v) = 0 \) is also true regardless of the values of the \( c_i \)'s.
2. (a) [12 points] Find the general solution for $3u_t + 4u_x = 0$. If we add in the condition that $u(x, 0) = e^{-x^2}$, is the resulting solution unique? Explain how this examples illustrates why the PDE is referred to as a “simple transport equation”, and give the transport speed.

$$3u_t + 4u_x = 0 \iff \begin{align*} \frac{dx}{dt} &= \frac{4}{3} \\
\frac{du}{dt} &= 0 \end{align*}$$

$\Rightarrow$ level curves have slope $\frac{dx}{dt} = \frac{4}{3}$ at every $x, t$.

$\Rightarrow$ the level curves are $x = \frac{4}{3} t + C$, or $x - \frac{4}{3} t = C$.

$\Rightarrow$ the general solution $u(x,t)$ is $u(x,t) = f(x-\frac{4}{3} t)$.

Now if $u(x,0) = e^{-x^2}$, $u(x,0) = f(x) = e^{-x^2}$ and so $u(x,t) = e^{-\left(x-\frac{4}{3} t\right)^2}$.

This illustrates why this is a “simple transport” problem because the solution is just the initial profile $e^{-x^2}$ translated to the right at speed $\frac{4}{3} = c$. Thus we can imagine it as the original concentration profile being transported by a fluid moving at constant speed $c = \frac{4}{3}$.

(b) [8 points] Is the following system of equations well-posed? Explain.

$$u_x + 2xy^2 u_y = 0$$

$u(x, 0) = \phi(x)$

Solving, we have:

$$\begin{align*}
\langle 1, 2xy^2 \rangle \cdot \langle u_x, u_y \rangle &= 0 \\
\Rightarrow \frac{dy}{dx} &= 2xy^2 & \text{for level curves of } u.
\end{align*}$$

$$\Rightarrow \int \frac{1}{y^2} dy = \int 2x dx$$

for level curves of $u$.

$$\Rightarrow -\frac{1}{y} = x^2 + C$$

(i.e., $-\frac{1}{y} + x^2 = C$)

$\Rightarrow$ the general solution has the form

$$u(x,y) = f\left(-\frac{1}{y} + x^2\right)$$

When we try to apply the “initial” condition:

$$u(x,0) = \phi(x) = f\left(-\frac{1}{y} + x^2\right)$$

we see that we cannot.

Thus we cannot show existence of a solution for this problem = ill-posed.
3. Consider the PDE
\[ u_{xx} - 4u_{xy} - 4u_{yy} = 0 \]

(a) [6 points] Classify the PDE as either elliptic, hyperbolic, or parabolic.

(b) [14 points] Choose a change of variables and use it to see the PDE as one of the wave equation/heat equation/Laplace's equation in terms of those new variables.

\[ a_{11} = 1 \quad a_{12} = -2 \quad a_{22} = -4 \]
\[ a_{12} = -2 \quad \Rightarrow \quad a_{12}^2 = 4 \quad \Rightarrow \quad a_{12}^2 > a_{11}a_{22} \]

so the PDE is hyperbolic.

Since the PDE is hyperbolic, we know there is some change of variables that will transform it into a PDE like the wave equation.

Try:
\[ x = a_{11} \int = \int \]
\[ y = a_{12} \int + \sqrt{a_{22} - a_{11}a_{22}} \int = -2 \int + \sqrt{8} \int \]

\[ \Rightarrow \frac{\partial}{\partial \int} = \frac{\partial}{\partial x} \int + \frac{\partial}{\partial y} \int = \frac{\partial}{\partial x} - 2 \frac{\partial}{\partial y} \]

\[ \Rightarrow \frac{\partial}{\partial \int} = \frac{\partial}{\partial x} \int + \frac{\partial}{\partial y} \int = \sqrt{8} \frac{\partial}{\partial y} \]

"Factorizing" our PDE by completing the square:
\[ 0 = \left( \frac{1}{2} \frac{\partial^2}{\partial x^2} - 4 \frac{\partial^2}{\partial x \partial y} - 4 \frac{\partial^2}{\partial y^2} \right) u = \left( \frac{\partial^2}{\partial x^2} - 2 \frac{\partial^2}{\partial x \partial y} + 4 \frac{\partial^2}{\partial y^2} \right) u - 8 \frac{\partial^2}{\partial y^2} u \]
\[ = \left( \frac{1}{2} \frac{\partial}{\partial x} - 2 \frac{\partial}{\partial y} \right)^2 u - 8 \frac{\partial^2}{\partial y^2} u \]
\[ = \left( \frac{\partial}{\partial \int} \right)^2 u - \left( \frac{\partial}{\partial \int} \right)^2 u = u_{yy} - u_{\eta \eta} \]

\[ \Rightarrow \text{the wave again in } x, \eta. \]
4. (a) [7 points] Suppose that there is a function $u(x, t)$ such that $u_t = 3u_{xx}$ and that:

$$u(x, 0) = 1 - x \quad u_x(0, t) = 0 \quad u(1, t) = 0$$

Describe fully one physical situation that could be modeled by this IBVP, and be sure to include what $u(x, t)$ represents as part of this description.

This could model the spread of heat in a uniform metal rod of length 1, where $u(x, t)$ = temperature of the rod at position $x$, time $t$.

Then the initial temp. distribution would be described by $1-x$; the end at $x=0$ would be perfectly insulated, and the temp at $x=1$ would be held at zero degrees.

(b) [8 points] Assuming the time-dependent setup from part (a), can the physical system come to a steady state solution? If so, what is it and how is the result interpreted in the physical application? Does it match your physical intuition of the situation? If it cannot come to steady-state, why not?

Physical Intuition:
Since one end is perfectly insulated and the other is held at 10°, I expect the only solution unchanging over time to be a line 10° over the whole rod.

Otherwise, if for example there is a region on the rod colder than 10°, the heat will conduct down the rod from the 10° end to warm it up... Similarly, if there is a hotter region...

The steady state is given by:

$$0 = 3u_{xx}$$

$$u(0) = 0 \quad u(1) = 0$$

$$u_x(0) = 0 \Rightarrow u(x) = A x + B.$$  
$$u'(0) = 0 \Rightarrow A = 0.$$  
Thus $u(x) = B.$

If $u(1) = 10 \Rightarrow u(1) = B = 10 \Rightarrow u(x) = 10$ for all $x$, on the temp.

the rod in a constant 10 degree.

(c) [5 points] Suppose that you have a 10 inch long string undergoing small vibrations in a plane, one end of which is pinned down and the other end is begin moved by a machine so that it's height is given by $sin(2\pi t)$ at any time $t$. If the string is initially at equilibrium, set up the IBVP that models this situation.

$$U_{tt} = c^2 U_{xx}$$
$$U(0, t) = 0$$
$$U(10, t) = \sin(2\pi t)$$
$$U(x, 0) = 0$$
5. [20 points] Choose one:

(a) Derive the wave equation from first principles. Include explanation for steps AND all assumptions made.

OR

(b) Derive the diffusion equation from first principles. Include explanation for steps AND all assumptions made.

**Wave equation:**

Suppose we have a perfectly flexible elastic string vibrating in a single plane (transversely only) such that the amplitude of the vibration is very small. Let \( u(x,t) \) = displacement from equilibrium at position \( x \) and time \( t \).

Let \( T(x,t) \) be the tension of the string at position \( x \) and time \( t \).

Since it's flexible, elastic → the tension will be tangent to the string itself,

\[
T(x,t) = \frac{||\langle 1, u_x \rangle||}{\sqrt{1 + (u_x)^2}} \quad \text{since } \langle 1, u_x \rangle \text{ is also always tangent to the string.}
\]

The amplitude of the vibrations are small ⇒ \( u_x^2 \) is small

\[ 1 + u_x^2 \approx 1 \quad \Rightarrow \quad T \approx ||\langle 1, u_x \rangle|| \]

Since there is no vibration longitudinally, the net force in the \( x \) direction across any segment of string must be zero — so on the segment between \([x_0, x_1]\),

Since the force is due solely to the tension, we see

\[ \frac{2}{x_1 - x_0} \langle 1, u_x \rangle \bigg|_{x=x_0}^{x=x_1} = 0 \quad \Rightarrow \quad \text{the magnitude of the tension is the same at every point on the string, since } x_0, x_1 \text{ are arbitrary.}
\]

Let \( T = ||\langle 1, u_x \rangle|| \), so that \( T \) is constant.

Transversely, we have Newton's law \( F = ma \) gives

Net force in the \( y \)-direction on \([x_0, x_1]\)

\[ T u_x(x_1, t) - T u_x(x_0, t) = \int_{x_0}^{x_1} u_{yy} \, dx \]

Differentiating w.r.t. \( x \), gives:

\[ T u_{xx}(x, t) = u_{tt}(x, t) \]

If arbitrary ⇒ \( u_{tt} = T u_{xx} \) for all \( x, t \) or \( u_{tt} = c^2 u_{xx} \) for all \( x, t \) with \( c = \sqrt{T} \).
Suppose we have a uniform thin tube of length \( L \) (approximately 1D) containing a motionless fluid, within which there are particles of some substance suspended that move solely due to diffusion. Let \( \mu(x, t) \) = concentration (\( \text{mg/cm}^3 \)) at position \( x \), time \( t \) of the substance.

Then the total mass of substance between \( x_0 \) and \( x_1 \), is

\[
M(t) = \int_{x_0}^{x_1} \mu(x, t) \, dx.
\]

Thus the rate of change of the mass between \( x_0 + x \), is, at any time \( t \),

\[
\frac{dM}{dt} = \frac{d}{dt} \int_{x_0}^{x_1} \mu(x, t) \, dx = \int_{x_0}^{x_1} \mu'(x, t) \, dx.
\]  \hspace{1cm} (1)

Thinking physically, the only way the amount of mass between \( x_0 + x \), can change is if some of it diffuses in or out through an end (at \( x_0 \) or \( x_1 \)).

Fick's law says that the rate of diffusion at a point is proportional to the conc. gradient at that point, so the rate at which mass will cross each end is proportional to \( \mu' \), at that end.

There are 4 possible cases:

1. Mass diffuses in at \( x_0 \).
2. Mass diffuses out at \( x_1 \).
3. Mass diffuses in at \( x_1 \), out at \( x_0 \).
4. Mass diffuses out at \( x_0 \), in at \( x_1 \).

For case 1, if mass diffuses in at \( x_0 \), we get a positive contribution to \( dM/dt \) at this end (as mass inside is increasing). This occurs only if \( \mu_x(x_0, t) \leq 0 \) since particles will diffuse from regions of higher to lower conc., and at the other end since we lose mass at \( x_1 \), we have \( \mu_x(x_1, t) \leq 0 \) and this and gives a neg. contribution to \( dM/dt \). By Fick's law + these observations:

\[
\frac{dM}{dt} = \mu_x(x_1, t) - \mu_x(x_0, t)
\]
In fact for all \( \Theta \) cases, we have

\[
\frac{dM}{dt} = kU_x(x_{\Theta}, t) - kU_x(x_0, t).
\] (2)

Putting (1) and (2) together, we see

\[
\int_{x_0}^{x_{\Theta}} u_t(x, t) \, dx = kU_x(x_{\Theta}, t) - kU_x(x_0, t)
\]

and differentiating w.r.t. \( x_{\Theta} \),

\[
\Rightarrow u_t(x_{\Theta}, t) = kU_{xx}(x_{\Theta}, t).
\]

Since \( x_{\Theta} \) is arbitrary \( \Rightarrow u_t = kU_{xx} \) for all \( x, t \).