1. LORENTZIAN VECTOR SPACES.

\((V^{n+1},g)\) Lorentzian vector space: \(g\) is an inner product (non-degenerate symmetric bilinear form) with index 1, or signature \((-,+,+,...,+).\)

The main example is Minkowski space \(\mathbb{M}^{n+1}: \mathbb{R}^{n+1}\) with coordinates \(v = (x_0,x_1,...,x_n)\), metric:

\[\langle v,v \rangle_g = -x_0^2 + x_1^2 + \ldots + x_n^2.\]

By convention, the light cone \(\Lambda\) does not include the origin:

\[\Lambda = \{v \in V; \langle v,v \rangle_g = 0, v \neq 0 \} = \lambda_+ \cup \lambda_- .\]

An “orthonormal basis” \(\{e_0,e_1,...,e_n\}\) satisfies:

\[\langle e_0,e_0 \rangle = -1, \quad \langle e_i,e_j \rangle = 0, \quad \langle e_i,e_0 \rangle = 0.\]

Lorentz causal character. ([ON, p.140]. A vector \(v \in V\) is spacelike if \(\langle v,v \rangle \geq 0\); null if \(v \in \Lambda\) and timelike if \(\langle v,v \rangle < 0\).

A subspace \(W \subset V\) is spacelike if \(g|_W\) is positive-definite; timelike (or Lorentzian) if \(g|_W\) is non-degenerate with index 1, degenerate if \(g|_W\) is degenerate. (Remark: \(V\) admits a degenerate subspace iff \(\dim V \geq 3\); then \(\text{span}\{e_0 + e_1, e_2\}\) is degenerate.) These are exhaustive possibilities, characterized by the following three lemmas.

**Lemma 1.** \(v \in V\) timelike \(\Rightarrow v^\perp \subset V\) spacelike, and \(V = \mathbb{R}v \oplus v^\perp\).

More generally, \(W \subset V\) timelike \(\Rightarrow W^\perp \subset V\) spacelike, and conversely (recall \((W^\perp)^\perp = W\).)

**Proof.** Recall a subspace \(W \subset V\) is non-degenerate iff \(W \oplus W^\perp = V\).

\(\mathbb{R}v\) is non-deg. with index 1, hence \(V = \mathbb{R}v \oplus v^\perp\) and \(v^\perp\) is non-degenerate. Thus \(\text{ind}(V) = \text{ind}(\mathbb{R}v) + \text{ind}(v^\perp)\), so \(\text{ind}(V^\perp) = 0\) and \(V^\perp\) is spacelike.

**Lemma 2.** Let \(V\) be Lorentzian, \(W \subset V\), \(\dim W \geq 2\). TFAE: (1) \(W\) is Lorentzian; (2) \(W\) contains two l.i. null vectors; (3) \(W\) contains a timelike vector.

**Proof.** (1) \(\Rightarrow\) (2): If \(\{e_0,e_1\}\) is orthonormal, \(e_0 \pm e_1\) are null and l.i. (2) \(\Rightarrow\) (3): we claim that if \(u,v\) are null and l.i., then \(\langle u,v \rangle \neq 0\). Assuming this, since \(\langle u \pm v,u \pm v \rangle = \pm 2\langle u,v \rangle\), one of \(u \pm v\) must be timelike.

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To prove the claim, write \( u = a_0 e_0 + u_s, v = b_0 e_0 + v_s \) with \( u_s, v_s \) spacelike, and use \( \langle u, v \rangle = 0 \) and equality in Cauchy-Schwarz to conclude \( u_s = \lambda v_s, a_0 = \pm \lambda b_0 \). With the + sign we contradict linear independence, with the - sign we get \( u - \lambda v = -2\lambda b_0 e_0 \), contradicting \( u - \lambda v \) null.

(3) \( \Rightarrow \) (1): \( v \in W \) timelike \( \Rightarrow W^\perp \subset v^\perp \) spacelike \( \Rightarrow W = (W^\perp)^\perp \) Lorentzian.

**Lemma 3.** Let \( V \) be Lorentzian, \( W \subset V \). TFAE: (1) \( W \) is degenerate; (2) \( W \) contains a null vector, but not a timelike vector; (3) \( W \cap \Lambda = L \setminus \{0\} \), where \( L \subset V \) is a one-dimensional subspace. *(FIGURE.) [ON p. 142]*

Proof. (1)\( \Rightarrow \) (2): \( W \) contains a null vector, and by lemma 2 not a timelike vector. (2)\( \Rightarrow \) (3): \( W \cap \Lambda \) is non-empty, and cannot contain 2 l.i vectors (by Lemma 2); (3) \( \Rightarrow \) (1): \( W \) can’t be spacelike or timelike, so it must be degenerate.

**Timecones.** Let \( T \) be the set of timelike vectors in a Lorentzian vector space \( V \). For \( u \in T \) the open set:

\[
C(u) = \{ v \in T ; \langle v, u \rangle < 0 \}
\]

is the *timecone* in \( V \) containing \( u \). The opposite timecone is \( C(-u) = -C(u) \).

**Lemma 4.** Timelike vectors \( v, w \in V \) are in the same timecone \( C(u) \) iff \( \langle v, w \rangle < 0 \).

Proof. See [ON p.143] It follows that \( u \in C(v) \) iff \( v \in C(u) \).

For \( v \in V \) Lorentzian, we let \( |v| = |g(v, v)|^{1/2} \). (Note: this is not a norm.)

**Proposition 1.** Let \( v, w \) be timelike vectors in \( V \) Lorentzian.

(i) (Reverse Cauchy-Schwartz): \( |\langle v, w \rangle| \geq |v||w| \), with equality iff \( v, w \) are collinear.

(ii) (Reverse triangle inequality): if \( v, w \) are timelike and in the same timecone, \( |v + w| \geq |v||w| \), with equality iff \( v, w \) are collinear.

Proof. [ON p. 143]

**Hyperquadrics** in a Lorentzian vector space. In \( \mathbb{M}^{n+1}, n \geq 2 \), consider the quadratic form:

\[
q(u) = -u_0^2 + u_1^2 + \ldots + u_n^2.
\]

In addition to the light cone \( \Lambda = \{ u | q(u) = 0, u \neq 0 \} \), we have smooth, connected level sets (for \( r > 0 \)). The “pseudospheres” \( (n \text{-dimensional deSitter} \)
spacetime):

\[ S^n_1(r) = \{ u; q(u) = r^2 \}; \]

and the hyperbolic spaces:

\[ H^n(r) = \{ u; q(u) = -r^2, u_0 > 0 \}. \]

Proposition 2. (i) \( H^n(r) \) is Riemannian in the induced metric, and isometric to the unit ball \((\mathbb{D}^n, g)\) with the conformal metric:

\[ ds_g^2 = \frac{4r^2|du|^2}{1 - |u|^2}, \quad u \in \mathbb{D}^n = \{|u| < 1\} \]

(ii) \( S^n_1(r) \) is Lorentzian in the induced metric, and isometric to the product \((\mathbb{S}^{n-1} \times \mathbb{R}, g)\), with the “skew-product metric”:

\[ ds_g^2 = -dt^2 + r^2 \cosh^2(\frac{t}{r})|du|^2, \quad (u, t) \in \mathbb{S}^{n-1} \times \mathbb{R}. \]

Proof. Note that for both level sets \( M^n \), we have:

\[ T_u M^n = \{ v \in M^{n+1}; q(v, u) = \langle v, u \rangle = 0 \} = u^\perp. \]

In the case of \( u \in H^n(r) \), \( u \) is timelike, so \( u^\perp \) is spacelike; hence the induced metric is Riemannian. In the case of \( u \in S^n_1(r) \), \( u \) is spacelike, so \( u^\perp \) is timelike; hence the induced metric is Lorentzian.

(i) Consider the diffeomorphism given by “stereographic projection from \(-re_0\)”:

\[ \mathbb{D}^n \to H^n(r), \quad u \mapsto (x, x_0) \in \mathbb{R}^n \times \mathbb{R}, \quad x = \frac{2ru}{1 - |u|^2}, x_0 = \frac{1 + |u|^2}{1 - |u|^2} > 0. \]

(Note \( x = \frac{2r}{r} u \).) Direct calculation yields (exercise):

\[ |dx|^2 = \frac{4r^2}{(1 - |u|^2)^2} [(1 - |u|^2)^2|du|^2 + 4(u \cdot du)^2], \quad dx_0^2 = \frac{16r^2}{(1 - |u|^2)^2} (u \cdot du)^2, \]

so we obtain the claimed expression for \( |dx|^2 - dx_0^2 \).

(ii) In the case of \( S^n_1(r) \), consider the diffeomorphism:

\[ S^{n-1} \times \mathbb{R} \to S^n(r), |u| = 1, \quad (u, t) \mapsto (x, x_0), \quad x = r(\cosh(\frac{t}{r}))u, x_0 = r \sinh(\frac{t}{r}). \]

A simple calculation (using \( u \cdot du = 0 \)) yields the claimed expression for \( |dx|^2 - dx_0^2 \).