1. Geodesic deviation: the curvature tensor and Jacobi fields.

Let \((M, g)\) be a Riemannian manifold, \(p \in M\). Suppose we want to measure the “instantaneous spreading rate” of geodesic rays issuing from \(p\). The natural way to do this is to consider a geodesic variation:

\[
f(t, s) = \exp_p(tv(s)), \quad v(s) \in T_p M, \quad v(0) = v, v'(0) = w.
\]

Then the “spreading rate” is measured by the variation vector field \(V(t)\):

\[
V(t) = \frac{\partial f}{\partial s |_{s=0}} = d\exp_p(tv)[tw],
\]

a vector field along the geodesic \(\gamma(t) = \exp_p(tv)\). Let’s try to find a differential equation satisfied by \(V\). We have:

\[
\frac{DV}{dt} = D\frac{\partial f}{dt} = \frac{D\partial f}{ds} \frac{d\exp_p(tv)}{dt} = D\frac{D\partial f}{ds \partial t}.
\]

Setting \(W = \frac{\partial f}{dt}\) (a vector field along \(f\)), we see that \(DW/dt \equiv 0\):

\[
\frac{D}{dt}\left(\frac{\partial f}{dt}\right) = \frac{D}{dt}(d\exp_p(tv)[v(s)]) = \frac{D}{dt}\gamma_s(t) = 0,
\]

since \(\gamma_s(t) = \exp_p(tv(s))\) is a geodesic \((\gamma_s(0) = p, \dot{\gamma}_s(0) = v)\). Thus we need to compute the vector field \(X(t)\) along \(f\):

\[
X(t) := \frac{D}{dt}DW - \frac{D}{dt}DW,
\]

for then, along \(\gamma(t)\):

\[
\frac{D^2V}{dt^2} = X(t).
\]

We compute in a coordinate chart:

\[
f(s, t) = (x^i(s, t)) \in \mathbb{R}^n, \quad W(s, t) = a^i(s, t)\partial_{x^i}.
\]

Using the symmetry of the connection, we find:

\[
X = a^i \partial_{x^k} \partial_{x^j} \partial_{x^i} (\nabla_{\partial_{x^k}} \nabla_{\partial_{x^j}} \partial_{x^i} - \nabla_{\partial_{x^j}} \nabla_{\partial_{x^k}} \partial_{x^i}).
\]

We now consider the linearity over smooth functions of this commutator of covariant derivatives. We find:

\[
\nabla_{\partial_{x_k}} \nabla_{\partial_{x_j}} (f \partial_{x_i}) - \nabla_{\partial_{x_j}} \nabla_{\partial_{x_k}} (f \partial_{x_i}) = f(\nabla_{\partial_{x_k}} \nabla_{\partial_{x_j}} \partial_{x_i} - \nabla_{\partial_{x_j}} \nabla_{\partial_{x_k}} \partial_{x_i}),
\]
and, assuming we have normal coordinates at \( p \) (so \( \nabla_{\partial x_k} \partial_{x_i} = 0 \) at \( p \)):

\[
\nabla_{f\partial x_k} \nabla_{\partial x_j} \partial_{x_i} - \nabla_{\partial x_j} \nabla_{f\partial x_k} \partial_{x_i} = f(\nabla_{\partial x_k} \nabla_{\partial x_j} \partial_{x_i} - \nabla_{\partial x_j} \nabla_{\partial x_k} \partial_{x_i}).
\]

Thus, by linearity, we have:

\[
X(t) = \nabla_{\partial_t f} \nabla_{\partial_s f} W - \nabla_{\partial_s f} \nabla_{\partial_t f} W.
\]

This suggests considering, given three vector fields \( X,Y,W \), the vector field:

\[
\nabla_X \nabla_Y W - \nabla_Y \nabla_X W.
\]

A natural question is whether this is “tensorial” (linear over smooth functions) in each of \( X,Y,W \). Starting with \( W \), we find:

\[
(\nabla_X \nabla_Y - \nabla_Y \nabla_X)(fW) = f(\nabla_X \nabla_Y - \nabla_Y \nabla_X)W + [X,Y]f.
\]

This suggests subtracting the term \( \nabla_{[X,Y]}W \). Computing again:

\[
(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})(fW) = f(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})W.
\]

This motivates the definition:

**Definition.** The \( (3,1) \)-Riemann curvature tensor \( R \) assigns to three vector fields \( (X,Y,Z) \) on \( M \) the vector field:

\[
R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z.
\]

**Exercise.** This assignment is linear over smooth functions in each of \( X,Y \) and \( Z \).

For a vector field \( W(t,s) \) along an immersion \( f(t,s) \), we get the Ricci equation:

\[
\frac{D}{dt} \frac{D}{ds} W - \frac{D}{ds} \frac{D}{dt} W = R(\partial_t f, \partial_s f) W.
\]

Returning to the vector field \( W(t,s) = \partial_t f \) along the geodesic variation \( f(t,s) \) (where the curves \( t \mapsto f(t,s) \) are geodesics), we find:

\[
\frac{D^2 V}{dt^2} = \frac{D}{dt} \frac{D}{ds} W = R(\partial_t f, \partial_s f) \partial_t f,
\]

and at \( s = 0 \) (since \( \partial_t f|_{s=0} = \dot{\gamma} \) and \( \partial_s f|_{s=0} = V \)):

\[
\frac{D^2 V}{dt^2} + R(V, \dot{\gamma}) \dot{\gamma} = 0.
\]
This is the Jacobi equation for the “geodesic deviation” vector field $V(t)$; its solutions are Jacobi fields along $\gamma(t)$.

Remark. To find the first-order initial condition for $V(t)$, consider:

$$
\frac{DV}{dt} \mid_{t=0} = \frac{D}{dt} \frac{df}{ds} \mid_{t=0, s=0} = \frac{D}{ds} \frac{df}{dt} \mid_{t=0, s=0} = \frac{d}{ds} \gamma_s(0) = v'(0) = w.
$$

We conclude:

$J(t) = d\exp_p(tv)\{tw\}$ is the Jacobi field along $\gamma(t) = \exp_p(tv)$ with IC $J(0) = 0, \dot{J}(0) = w$.

In particular: $d\exp_p(v)\{w\} = J(1)$. This expresses the differential of the exponential map in terms of the solution of a differential equation along $\gamma(t)$.

2. The case of graphs in euclidean space. Consider the surface $M \subset \mathbb{R}^{n+1}$:

$$
M = \text{graph}(F) = \{(x, F(x)); x \in \mathbb{R}^n\}, \quad F : \mathbb{R}^n \to \mathbb{R}, \quad \nabla F(0) = 0.
$$

The induced metric and inverse metric tensors are (check!):

$$
g_{ij} = \delta_{ij} + F_i F_j, \quad g^{ij} = \delta^{ij} - \frac{F_i F_j}{1 + |\nabla F|^2}.
$$

Given the assumptions made at $x = 0$, we find:

$$
g_{ij}\mid_{l}(0) = 0 \text{ and hence } \Gamma^k_{ij}(0) = 0.
$$

For the first derivatives of the Christoffel symbols at $x = 0$:

$$
\partial_x \Gamma^k_{ij}(0) = \frac{1}{2} (g_{ik|j|m} + g_{jk|i|m} - g_{ij|k|m}),
$$

and for the curvature tensor$^1$:

$$
R(\partial_{x_j}, \partial_{x_k})\partial_{x_l} = \nabla_{x_j} \nabla_{x_k} \partial_{x_l} - \nabla_{x_k} \nabla_{x_j} \partial_{x_l}
$$

$$
= (\Gamma^l_{ik|j} - \Gamma^l_{ij|k} + \Gamma^m_{ik} \Gamma^l_{jm} - \Gamma^m_{ij} \Gamma^l_{km}) \partial_{x_l}
$$

$$
:= R^l_{ijk} \partial_{x_l}.
$$

$^1$Note here we’re using the notation $R$ for what is in fact the pull-back $\varphi^* R$ of the (3,1) curvature tensor under the graph chart $\varphi(x) = (x, F(x))$.
Thus at $x = 0$ we have:

$$R^l_{jki}(0) = \Gamma^l_{ikj} - \Gamma^l_{ijk}$$

Suppose we choose the axes so that the Hessian of $F$ is diagonal at $x = 0$:

$$Hess(F)|_0 = \text{diag}(\lambda_1, \ldots, \lambda_n), \quad F_{ij}(0) = \lambda_i \delta_{ij} \quad \text{(no sum)}.$$ 

Then, since $g_{ij|kl}(0) = F_{ik}F_{jl}(0) + F_{il}F_{jk}(0) = \lambda_i \lambda_j (\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il})$, the only non-zero second derivatives of the metric at $x = 0$ are:

$$g_{ii|i} = 2\lambda_i^2, \quad g_{ij|ij} = g_{ij|ji} = \lambda_i \lambda_j \quad (i \neq j).$$

(in particular, $g_{ij|jj} = 0$ for $i \neq j$.) Thus the only potentially non-vanishing first-order derivatives of Christoffel symbols (at $x = 0$) have either all four indices equal, or two pairs of equal indices:

$$\Gamma^i_{ii|i} = \lambda_i^2,$n

$$\Gamma^i_{ij|j} = \frac{1}{2} (g_{ii|jj} + g_{ij|ij} - g_{ij|ij}) = 0 \quad (i \neq j),$$

$$\Gamma^j_{ij|i} = \frac{1}{2} (g_{ij|ii} + g_{ij|ij} - g_{ij|ij}) = \lambda_i \lambda_j \quad (i \neq j).$$

We conclude:

$$\Gamma^l_{ik|j} = \lambda_i \lambda_j \delta_{ik}\delta_{lj},$$

and for the components of the curvature tensor at $x = 0$:

$$R^l_{jki} = \Gamma^l_{ikj} - \Gamma^l_{ijk} = \lambda_j \lambda_k (\delta_{ik}\delta_{lj} - \delta_{ij}\delta_{lk}) \quad \text{(no sum)}.$$ 

For the components of the $(4,0)$-curvature tensor:

$$R_{jkim} := \langle R(\partial_{x_j}, \partial_{x_k})\partial_{x_i}, \partial_{x_m} \rangle,$$

using the fact that $g_{ij}(0) = \delta_{ij}$ we find, at $x = 0$:

$$R_{jkim} = \lambda_j \lambda_k (\delta_{ik}\delta_{jm} - \delta_{ij}\delta_{km})$$

$$= D^2F(\partial_{x_i}, \partial_{x_k})D^2F(\partial_{x_j}, \partial_{x_m}) - D^2F(\partial_{x_i}, \partial_{x_j})D^2F(\partial_{x_k}, \partial_{x_m}),$$

where $D^2F$ is the Hessian quadratic form of $F$. 

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By linearity, we have for arbitrary vector fields $X,Y,Z,W$ on $\mathbb{R}^n$, and at $x = 0$:

$$
\langle (\varphi^* R)(X,Y)Z,W \rangle_g = D^2 F(Z,Y)D^2 F(X,W) - D^2 F(Z,X)D^2 F(Y,W).
$$

Since both sides of this equation are “tensorial” (4-linear over functions), it in fact holds everywhere, and expresses the $(4,0)$ curvature tensor in terms of the Hessian of $F$.

Kulkarni-Nomizu product. Given two quadratic forms $Q, \bar{Q}$ (i.e., symmetric bilinear forms) in a vector space $E$, their Kulkarni-Nomizu product is the 4-linear form on $E$:

$$(Q \odot \bar{Q})(x,y,z,w) := \frac{1}{2} [Q(x,z)\bar{Q}(y,w) - Q(y,z)\bar{Q}(x,w) + \bar{Q}(x,z)Q(y,w) - \bar{Q}(y,z)Q(x,w)].$$

Exercise. (i) $Q \odot \bar{Q}$ has the same algebraic symmetries as the $(4,0)$-Riemann curvature tensor, except for the first Bianchi identity: it is skew-symmetric in $(x,y)$, skew-symmetric in $(z,w)$ and symmetric under swapping the ordered pairs $(x,y)$ and $(z,w)$. Thus $Q \odot \bar{Q}$ is a quadratic form in the space of alternating 2-vectors $\Lambda_2(E)$.

(ii) If $Q = \bar{Q}$, the 4-linear form $\omega = Q \odot Q$ also satisfies the algebraic Bianchi identity:

$$
\omega(x,y,z,w) + \omega(y,x,z,w) + \omega(z,y,x,w) = 0.
$$

In terms of the K-N product, the $(4,0)$-Riemann curvature tensor $Riem$ of a graph has the expression:

$$
Riem = -(D^2 F \circ D\pi) \odot (D^2 F \circ D\pi),
$$

where $D\pi(p) : T_pM \rightarrow \mathbb{R}^n$ is the inverse of the differential graph chart $D\varphi$, and

$$(D^2 F \circ D\pi)(X,Y) := D^2 F(D\pi[X],D\pi[Y]), \quad X,Y \in T_pM$$

Recall that if $E$ has an inner product, there is an associated inner product in $\Lambda_2(E)$ uniquely determined by:

$$
\langle x \wedge y, z \wedge w \rangle = \langle x, z \rangle \langle y, w \rangle - \langle y, z \rangle \langle x, w \rangle
$$

$^2$Here we revert to more precise notation: $\varphi^* R$ is a $(3,1)$ tensor on $\mathbb{R}^n$, the pullback of the curvature tensor under the graph chart $\varphi$. 

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(which itself has the structure of a K-N product!) Note, in particular:

\[ |x \wedge y|^2 = |x|^2|y|^2 - \langle x, y \rangle^2. \]

Thus, given a quadratic form on \( \Lambda_2(E) \) we have an associated symmetric linear operator from \( \Lambda_2(E) \) to \( \Lambda_2(E) \). In the case of the \((4,0)\)-Riemann curvature tensor \( \text{Riem} \), the associated symmetric linear operator (at each \( p \in M \))

\[ \mathcal{R}_p : \Lambda_2(T_pM) \to \Lambda_2(T_pM) \]

is known as the curvature operator, and has found important applications in recent years.

The two-dimensional case. If \( n = 2 \), the only non-zero components of \( \text{Riem} \) (at \( p \in M \)) have the form \( \langle R(X,Y)X,Y \rangle \), with \( X,Y \) linearly independent (in \( T_pM \)). If \( M \) is the graph of \( F \):

\[ \langle R(X,Y)X,Y \rangle = -(D^2F \circ D^2F)(D\pi X, D\pi Y, D\pi X, D\pi Y). \]

If the \( \{e_1, e_2\} \) is an orthonormal basis of \((\mathbb{R}^n, g)\) diagonalizing \( D^2F \) with eigenvalues \( \lambda_1, \lambda_2 \), we find:

\[ (D^2F \circ D^2F)(e_1, e_2, e_1, e_2) = D^2F(e_1, e_1)D^2F(e_2, e_2) - (D^2F(e_1, e_2))^2 = \lambda_1\lambda_2. \]

This motivates the definition, for general dimensions \( n \) (changing the order of the second pair \( X,Y \) to get rid of the sign):

\[ \text{Definition.} \quad \text{Let} \ v, w \in T_pM \text{ be linearly independent. The sectional curvature of } M \text{ along the 2-dimensional subspace } E \subset T_pM \text{ spanned by } v \text{ and } w \text{ is the real number } \sigma_E(p) \text{ defined by:} \\
\[ \langle R(X,Y)X,Y \rangle(p) = \sigma_E(p)|v \wedge w|_{g_p}^2, \]

where \( X, Y \) are vector fields on \( M \) with \( X(p) = v, Y(p) = w \) and \( |v \wedge w|_{g_p}^2 = |v|_{g_p}^2|w|_{g_p}^2 - \langle v, w \rangle_{g_p}^2 \).

\[ \text{Exercise.} \quad \sigma_E(p) \text{ depends only on the two-dimensional subspace } E, \text{ not on the choice of basis.} \]

Thus, in the two-dimensional case (for the graph of a function \( F \)), the sectional curvature \( \sigma \) is the product of the eigenvalues of the Hessian \( D^2F \) ( \( \sigma = \lambda_1\lambda_2 \)), and explicitly determines the \((4,0)\)-Riemann curvature tensor, via:

\[ \langle R(X,Y)Z,W \rangle = -\sigma(\langle X,Z \rangle \langle Y,W \rangle - \langle Y,Z \rangle \langle X,W \rangle) = -\sigma \langle X \wedge Y, Z \wedge W \rangle. \]
It also follows that:

\[ \sigma > 0 \Leftrightarrow \lambda_1, \lambda_2 \text{ have the same sign}. \]

3. Hypersurfaces in euclidean space. Let \( M^n \subset \mathbb{R}^{n+1} \) be a sub-manifold. Given \( p \in M \), we define a local graph chart at \( p \), \( \varphi : U \to M \), \( U \subset \mathbb{R}^n \) open, via:

\[ \varphi(x) = (x, F(x)) \in U \times \mathbb{R}, \quad F : U \to \mathbb{R} \text{ smooth}, \quad (0, F(0)) = p, \nabla F(0) = 0. \]

The “upward” unit normal in a neighborhood of \( p \) is given in this chart by the map:

\[ N : U \to \mathbb{R}^{n+1}, \quad N(x) = \frac{(-\nabla F(x), 1)}{\sqrt{1 + |\nabla F(x)|^2}}. \]

Thus if \( \hat{N} : M \to S^n \) denotes the Gauss map of \( M \), we have:

\[ \hat{N}(\varphi(x)) = N(x), \quad x \in U. \]

Vector fields \( X, Y \) in \( U \) correspond via \( \varphi \) to tangent vector fields \( \bar{X}, \bar{Y} \in \chi_M \):

\[ \bar{X} = D\varphi[X] = (X, \nabla F \cdot X), \quad \bar{Y} = D\varphi[Y] = (Y, \nabla F \cdot Y). \]

By direct computation at \( x = 0 \), we find:

\[ \langle D\hat{N}(0)[\bar{X}], \bar{Y} \rangle = -D^2 F(0)(X, Y). \]

And the chain rule gives: \( D\hat{N}(p)[\bar{X}] = D\hat{N}(p)D\varphi(0)[X] = DN(0)[X] \), so we find for the differential of the Gauss map:

\[ \langle D\hat{N}(p)[\bar{X}], \bar{Y} \rangle = -D^2 F(0)(X, Y) = -D^2 F(0)(D\pi(p)[\bar{X}], D\pi(p)[\bar{Y}]), \]

where \( D\pi(p) : T_p M \to \mathbb{R}^n, D\pi(p)[\bar{X}] = X \) if \( \bar{X} = (X, \nabla F \cdot X) \).

Note that since the last equality is “tensorial” (bilinear in \( \bar{X}, \bar{Y} \) over smooth functions), it in fact holds at all points of \( M \). In addition, it shows that the left-hand side is symmetric in \( (\bar{X}, \bar{Y}) \) (since the right-hand side is). This leads to the important definition of the second fundamental form (of \( M \) in \( \mathbb{R}^{n+1} \)), the quadratic form on \( TM \) given in terms of the Gauss map \( \hat{N} \) by:

\[ A(\bar{X}, \bar{Y}) := -\langle D\hat{N}[\bar{X}], \bar{Y} \rangle. \]
Remarks. (i) The “first fundamental form” is the induced metric. (ii) The purpose of the negative sign is to make the sectional curvature of the graph of a convex function positive (see below).

Above we established that, for a graph:

\[ A(\bar{X}, \bar{Y}) = D^2 F(D\pi[\bar{X}], D\pi[\bar{Y}]), \]

where \( D\pi \) is the inverse differential of the graph chart, \( D\phi \). Thus we have, for the (4,0)-curvature tensor of a hypersurface in \( \mathbb{R}^{n+1} \) the beautiful relation:

\[ Riem = -A \odot A. \]

To make this more concrete, consider a two-dimensional subspace \( E \subset T_pM \) which is invariant under the second fundamental form. This means \( S_p(E) \subset E \), where \( S_p : T_pM \to T_pM \) is the self-adjoint operator (with respect to the induced metric at \( p \in M \)) associated with \( A_p \). (Note \( S_p = -D\hat{N}(p) \)). Let \( \{e_1, e_2\} \) be an orthonormal basis of \( E \) diagonalizing the restriction \( S_p|_E \), with \( S_p(e_i) = \lambda_i e_i \) for \( i = 1, 2 \). We have for the sectional curvature of \( E \):

\[
\sigma_E = \langle R_p(e_1, e_2)e_2, e_1 \rangle = A_p \odot A_p(e_1, e_2, e_1, e_2) \\
= A_p(e_1, e_1)A_p(e_2, e_2) - (A_p(e_1, e_2))^2 = \lambda_1 \lambda_2 = \det(S_p|_E).
\]

This is an important conclusion: if \( E \) is a two-dimensional subspace of \( T_pM \) invariant under the “shape operator” \( S_p = -D\hat{N}(p) \) at a point \( p \in M \), the sectional curvature at \( p \) along \( E \) (which depends only on the first fundamental form and its derivatives up to second order) equals the determinant of the restriction of the shape operator to \( E \) (which seems to depend on the second fundamental form, or on the unit normal and its first derivative).

In two dimensions, the “invariance” condition is unnecessary. The eigenvalues of \( S_p \) are the “principal curvatures” at \( p \in M \), and their product is the Gauss curvature \( K = \lambda_1 \lambda_2 \) at \( p \). We conclude:

Gauss’s Teorema Egregium: \( \sigma = K. \)

The fact that the sectional curvature equals the Gauss curvature is surprising since \( \sigma \) depends only on the induced metric (tangential information), while \( K \) seems to depend on the embedding of the surface in \( \mathbb{R}^3 \) (specifically, on how the unit normal “turns” near \( p \)).
Recall also that in two dimensions the Gauss curvature of a hypersurface equals the Jacobian of the Gauss map, so we have:
\[ \sigma(p) = K(p) = \det D\hat{N}(p), \quad p \in M. \]

4. The differential Bianchi identity.

Theorem. The \((3,1)\) curvature tensor satisfies:

Proof. We compute in a geodesic frame at \(P\), so that \(\nabla_X Y(p) = 0\) for all vector fields \(X, Y\). Then the left-hand side is, at \(p\) (using \([X, Y](p) = 0\)):
\[
\]
\[
= ([\nabla_X, [\nabla_Y, \nabla_Z]](W) + [\nabla_Y, [\nabla_Z, \nabla_X]](W) + [\nabla_Z, [\nabla_X, \nabla_Y]](W),
\]
and the fact this vanishes follows from the Jacobi identity for commutators of linear operators.

An important corollary is the \textit{contracted Bianchi identity}, which is useful in General Relativity. It states:
\[
\text{div}(\text{Ric}) - \frac{1}{2} \nabla S = 0,
\]
where \(\text{Ric}\) and \(S\) are the \((1,1)\) Ricci tensor and the scalar curvature, and \(\text{div}(\text{Ric}) = \sum_i \nabla_{e_i}(\text{Ric})(e_i)\).

To see this, compute in an orthonormal frame which is geodesic at \(p\) (so \(\nabla_{e_i} e_j(p) = 0\)). Then at \(p\):
\[
\langle \text{div}(\text{Ric}), X \rangle = \sum_i \langle (\nabla_{e_i} \text{Ric})e_j, X \rangle = \sum_{i,j} \langle (\nabla_{e_j} R)(e_j, e_i)e_i, X \rangle = \sum_{i,j} \langle (\nabla_{e_j} R)(e_i, X)e_j, e_i \rangle
\]
\[
= \sum_{i,j} [-(\nabla_{e_i} R)(X, e_j)e_j, e_i) - \langle (\nabla_X R)(e_j, e_i)e_j, e_i \rangle]
\]
(from the differential Bianchi identity just proved)
\[
= -\sum_j \langle (\nabla_{e_j} \text{Ric})(X), e_j \rangle - X(S) = -\langle \text{div}(\text{Ric}), X \rangle - X(S),
\]
proving the claim.

5. The Gauss and Codazzi equations for hypersurfaces.

Let $M \subset \bar{M}$ be a hypersurface (codimension 1-submanifold) with the Riemannian metric induced from $\bar{M}$. For vector fields $X \in \chi_M, Y \in \bar{\chi}_M$ (the space of vector fields on $\bar{M}$ restricted to $M$) we have the tangent-normal decomposition (with respect to a unit normal vector $N \in \bar{\chi}_M$):

$$\nabla_X Y = \nabla_X Y + A(X, Y)N,$$

where $A$ is the second fundamental form (recall $A(X, Y) = -\langle \bar{\nabla}_X N, Y \rangle$). Iterating this formula, we find:

$$\bar{\nabla}_X \bar{\nabla}_Y Z = \bar{\nabla}_X (\bar{\nabla}_Y Z + A(Y, Z)N) = \bar{\nabla}_X \bar{\nabla}_Y Z + X(A(Y, Z))N + A(Y, Z)\bar{\nabla}_X N.$$

Skew-symmetrising and taking tangential components (using $[X,Y](p) = 0$, for a frame geodesic at $p$):

$$\left[[\bar{R}(X,Y)Z\right]_{tan} = \bar{R}(X,Y)Z + A(Y,Z)\langle \bar{\nabla}_X N, W \rangle - A(X,Z)\langle \bar{\nabla}_Y N, W \rangle.$$

Taking inner product with $W \in \chi_M$, we find for the $(4,0)$ Riemann tensors:

$$\langle \bar{R}(X,Y)Z, W \rangle = \langle R(X,Y)Z, W \rangle + A(Y, Z)\langle \bar{\nabla}_X N, W \rangle - A(X,Z)\langle \bar{\nabla}_Y N, W \rangle$$


Using the Kulkarni-Nomizu product, we obtain for the $(4,0)$ curvature tensor of $M$ and $\bar{M}$ the relation:

$$\bar{Riem} = Riem + A \odot A.$$

This immediately implies, for the sectional curvatures along the 2-plane $span\{X,Y\}$:

$$\sigma_{XY} = \sigma_{XY} - A(X, X)A(Y, Y) + A(X, Y)^2 \quad \{X, Y\} \text{ orthonormal}$$

This is the general *Gauss equation* for hypersurfaces in a Riemannian manifold. On the other hand, taking inner product of the first relation above with the unit normal $N$:


This is the *Codazzi equation*. In terms of the shape operator:

$$(\nabla_X S)Y - (\nabla_Y S)X = -\bar{R}(X,Y)N.$$ 

In particular, $\langle (\nabla_X S)Y - (\nabla_Y S)X = 0$ if $\bar{M}$ is flat (or, more generally, of constant curvature.)