MATH 447, FALL 2016–Second Problem Set.

1. [F, p. 50] \( U \subset \mathbb{R}^m \) open, \( f : U \rightarrow \mathbb{R}^n \) function, \( x_0 \in U \). Show that \( f \) is continuous at \( x_0 \) if, and only if, for any sequence \( (x_n)_{n \geq 1} \) in \( U \) we have: 
\[
\lim_{n \to \infty} x_n = x_0 \Rightarrow \lim_{n \to \infty} f(x_n) = f(x_0).
\]

2. [F p. 62] Show that any closed subset of a compact topological space is compact.


7. Compactness implies the Finite Intersection Property. (In fact they are equivalent.)

4. (i) \( f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2 \) is not uniformly continuous on \( \mathbb{R} \) (in particular, not Lipschitz on \( \mathbb{R} \)); but \( f \) is locally Lipschitz on \( \mathbb{R} \).
(ii) \( f : (0, 1] \rightarrow \mathbb{R} \) given by \( f(x) = \sqrt{x} \) is locally Lipschitz.
(iii) \( f : [0, 1] \rightarrow \mathbb{R} \) given by \( f(x) = \sqrt{x} \) is not locally Lipschitz (in particular, not Lipschitz in \( [0, 1] \)).
(iv) \( f : \mathbb{R}_+ \rightarrow \mathbb{R} \) given by \( f(x) = \sqrt{x} \) is Hölder in \( \mathbb{R}_+ \), with exponent 1/2.

5. If \( f : \mathbb{R}^m \rightarrow \mathbb{R}^n \) is continuous and \( X \subset \mathbb{R}^m \) is bounded, the restriction \( f|_X : X \rightarrow \mathbb{R}^m \) is uniformly continuous.

6. The function \( f : \mathbb{R}^2 \rightarrow \mathbb{R}, f(x, y) = xy \) is not uniformly continuous. (Hint: Consider the sequences \( z_k = (k, 1/k) \) and \( w_k = (k, 0) \) in \( \mathbb{R}^2 \).)

7. The orthogonal \( n \times n \) matrices define a compact subset of \( \mathbb{R}^{n^2} \). (Recall an \( n \times n \) matrix \( A \) is orthogonal if \( A^T A = I_n \), where \( T \) denotes ‘transpose’ and \( I_n \) is the \( n \times n \) identity matrix.)

8. Let \( X \subset \mathbb{R}^{n+1} \setminus \{0\} \) a compact set continuing exactly one point on each half-line from \( 0 \in \mathbb{R}^{n+1} \). Show that \( X \) is homeomorphic to the unit sphere \( S^n \).

9. Let \( f : \mathbb{R}^m \subset \mathbb{R}^n \) be continuous. The following are equivalent:
   (i) \( \lim_{x \to \infty} f(x) = \infty \).
   (ii) For each compact subset \( K \subset \mathbb{R}^n \), we have that \( f^{-1}(K) \) is compact in \( \mathbb{R}^m \).
   (Such maps \( f \) are called proper.) If \( f : \mathbb{R}^m \rightarrow \mathbb{R}^n \) is proper and \( F \subset \mathbb{R}^m \) is closed, then \( f(F) \subset \mathbb{R}^n \) is also closed.

10. Any locally Lipschitz map \( f : K \rightarrow \mathbb{R}^n \) defined on a compact set \( K \subset \mathbb{R}^m \) is Lipschitz.