INSTRUCTIONS.

1. Solve each problem on a separate page and staple your pages. Include detailed solutions and derivations.

2. This is individual work! You may not consult anyone to solve this test, whether in person or over the internet.

3. Allowed sources: You may use results found in the texts by Weinberger or Strauss, the course handouts or your class notes (include their reference with page number, if you quote a result.) Internet sources are excluded. Plagiarized/copied solutions will result in no credit for the problem.

4. Interview. If I have reason to suspect a given student’s work was not done in accordance with 2. and 3. above, I will call the student for an interview, where he/she will be asked to explain the solutions to me. If a student is unable to explain how a solution was arrived at: no credit for the problem. If on two problems: grade ZERO on the test.
1. (i) Find the solution of the Dirichlet problem $\Delta u = 0$ on the disk of radius 1 in $\mathbb{R}^2$, with boundary data $h(\theta) = \cos^3(\theta)$. 

*Hint.* Use $\cos^3(\theta) = \frac{1}{4}(\cos 3\theta + 3 \cos \theta)$.

(ii) Now solve the exterior Dirichlet problem: find a harmonic function on the exterior of the unit disk in $\mathbb{R}^2$, with the same boundary data as in (i). Is the solution unique? (Explain why, or why not.)

2. (i) Find the most general solution to the exterior Neumann problem for the unit disk:

$$\Delta u = 0 \text{ in } \{r > 1\}; \quad u_r(1, \theta) = \sin(2\theta).$$

(There will be some arbitrary constants in your answer.)

(ii) Now show that there is only one solution if one adds the requirement $u(r, \theta) \to 0$ as $r \to \infty$.

3. Prove uniqueness for solutions of the ‘Robin problem’ (for a bounded domain $D \subset \mathbb{R}^n$):

$$\Delta u = f \text{ in } D, \quad \partial_n u + au = 0 \text{ on } \partial D.$$ 

(Here $a$ is a positive constant and $n$ denotes the outward unit normal.) *Hint:* Consider the difference of two solutions, and use Green’s first identity.

4. Use the method of image charges to find Green’s function and the Poisson kernel for the upper half-plane. (See the derivation for $\mathbb{R}^n$ in the notes.)

5. Find Green’s function for the upper half-ball of radius $a > 0$ in $\mathbb{R}^3$, with pole at an arbitrary point $x \in D$:

$$D = B^+_a = \{x \in \mathbb{R}^3; |x| < a, x_3 > 0\}.$$ 

*Hint:* Use Green’s function for the whole ball and reflection on the plane $x_3 = 0$.

6. (i) Use the reflection method to find Green’s function for the disk of radius $a$ in the plane, and then recover the Poisson kernel for the disk from it. (This is done in the notes for $\mathbb{R}^n, n \geq 3$).

(ii) Find a formula for the bounded solution of the Neumann problem in the upper half-plane:

$$\Delta u = 0 \text{ in } U = \{x = (x_1, x_2); x_2 > 0\} \quad \partial_{x_2} u = h(x) \text{ on } \{x_2 = 0\}.$$ 

*Hint:* Which boundary-value problem does the function $v = \partial_{x_2} u$ solve in $U$?