MATH 435, SPRING 2014-EXAM 2-4/9/2014 NAME:

1. Consider the sequence of functions in \([-1, 1] \):
   \[ f_n(x) = x^{2n}, n \geq 1. \]

   (i) Find a function \( g : [-1, 1] \rightarrow \mathbb{R} \) so that \( f_n \rightarrow g \) pointwise in \([-1, 1] \); sketch its graph and that of \( f_n \) (for some large \( n \)) in the same diagram.

   (ii) Does \( f_n \rightarrow g \) uniformly in \([-1/2, 1/2]\)? Does \( f_n \rightarrow g \) uniformly in \([-1, 1]\)? Justify (include the definition of uniform convergence).

   \[
   \begin{align*}
   (i) & \quad f_n(x) \rightarrow 0, \quad |x| < 1 \quad \text{for} \quad f_n(\pm 1) = 1, \\
   (ii) & \quad \max_{[-1/2, 1/2]} f_n(x) = \frac{1}{2^{2n}} \rightarrow 0, \\
   & \quad \max_{[-1, 1]} (f_n - g)(x) = 1 \quad \forall n \quad \text{but not in } [-1/2, 1/2],
   \end{align*}
   \]

2. Let \( u_0(x), \ x \in \mathbb{R}^n \), be a function with the property: \( \Delta u_0(x) \) is a linear function of the coordinates \( x_1, \ldots, x_n \).

   Find a function \( b(t) \) so that \( v(x, t) = u_0(x) + b(t)\Delta u_0(x) \) is a solution of the wave equation \( v_{tt} - \Delta v = 0 \) with initial conditions \( v(x, 0) = u_0(x), v_t(x, 0) = 0 \). Hint: Recall the Laplacian of a linear function is zero.

   \[
   \begin{align*}
   v_{tt} &= b''(t) \Delta u_0(x) \\
   \Delta v &= \Delta u_0(x) + b(t) \Delta (\Delta u_0(x)) = \Delta u_0(x) \\
   v(x, 0) &= u_0(x) + b(0) \Delta u_0(x) \rightarrow b(0) = 0 \\
   v_t(x, 0) &= b'(0) \Delta u_0(x) \rightarrow b'(0) = 0 \quad \text{if} \quad b(t) = \frac{t^2}{2}
   \end{align*}
   \]
3. Consider the functions $f, g$ in $[0, \pi]$ given below. Let $s^f_N, s^g_N$ be the partial sums of their Fourier cosine series.

(i) Sketch the graph of the appropriate $2\pi$-periodic extensions of $f$ and $g$, on the interval $[-2\pi, 2\pi]$. (The graphs in $[0, \pi]$ are given.)

(ii) Does $s^f_N \to f$ in $L^2[0, \pi]$? Justify.

(iii) Find the pointwise limits of $s^f_N(x)$ and $s^g_N(x)$, for each point $x \in [0, \pi]$.

(iv) For which of the functions $f, g$ does $s_N$ converge uniformly in $[0, \pi]$? Justify.

\[ f(x) = x^2 - \pi^2, \quad g(x) = 2x^3 - 3\pi x^2, \quad x \in [0, \pi]. \]
4. Find the value at the origin of the solution of the heat equation in \( \mathbb{R}^3 \), \( u_t - \Delta u = 0 \) (using standard coordinates \((x_1, x_2, x_3)\) in \( \mathbb{R}^3 \)), with initial condition:

\[
u_0(x_1, x_2, x_3) = \chi_{[-L, L]}(x_1) \chi_{D_R}(x_2, x_3)
\]

(that is, the product of the characteristic functions of the interval \([-L, L]\) in the \(x_1\)-axis and the characteristic function of the disk \(D_R = \{(x_2, x_3) : |x_2, x_3| < R\}\) in the \((x_2, x_3)\)-plane). Express your answer in terms of the functions \(erf_1\) and \(erf_2\), where:

\[
erf_n(\rho) = \frac{1}{\pi^{n/2}} \int_{\{|y| < \rho\}} e^{-|y|^2} dV_y, \text{ for } \rho > 0.
\]

\[
u_0(0, t) = y(0, t) w(0, t)
\]

\[
u(0, t) = \frac{1}{4\pi^2 t} \int_{\mathbb{R}^2} e^{-|y|^2/4t} \chi_{[-l, l]}(y) dy = \frac{1}{\sqrt{4\pi t}} \int_{[-l, l]} e^{-|y|^2/4t} dy
\]

\[
w(0, t) = \frac{1}{4\pi t} \int_{\mathbb{R}^2} e^{-|y|^2/4t} \chi_{D_R}(y) dy = \frac{1}{4\pi t} \int_{1y < R} e^{-|y|^2/4t} dA_y
\]

\[
p = \frac{\chi}{4\pi t} \int_{1p < R} e^{-|p|^2/4t} dA_p = erf_2\left(\frac{R}{\sqrt{4t}}\right)
\]

\[
u(0, t) = erf_1\left(\frac{L}{\sqrt{4t}}\right) erf_2\left(\frac{R}{\sqrt{4t}}\right)
\]
5. Consider the wave equation $u_{tt} - \Delta u = 0$ with initial conditions $u(x,0) = 0, u_t(x,0) = u_1(x)$, where $u_1(x) > 0$ everywhere in $B_R$, and $u_1 \equiv 0$ outside of $B_R$. $B_R = \{ x \in \mathbb{R}^n; |x| \leq R \}$.

(i) Write down the solution formulas for $u(x,t)$ in terms of $u_1 (x \in \mathbb{R}^n, t > 0)$, in the cases $n = 3$ and $n = 2$.

(ii) Draw spacetime diagrams in both cases ($n = 2, n = 3$), showing the regions in the $\{ r > 0, t > 0 \}$ quadrant where the solution is zero. (You may assume the solution is radial for each $t$, that is, depends only on $r = |x|$). Use the expressions in (i) to justify the answer.

(iii) Use the diagrams in (ii) to answer the question: what is the support of the solution at a given time $t_0$ (where $t_0 > R$) in the cases $n = 2$ and $n = 3$? The answer is a subset of $\mathbb{R}^n$ depending on $t_0$ and on $R$.

(Recall the support of $u$ at time $t_0$ is the set where $u(x,t_0) \neq 0$ (as a function of $x \in \mathbb{R}^n$, for fixed $t_0 > R$.)

\[
(i) \quad n = 3 \quad \mu(x,t) = \frac{i}{4\pi t} \int_{B_R \cap S^*_t(x)} \frac{\mu_1(y)}{4\pi} dA_y
\]

\[
(ii) \quad n = 2 \quad \mu(x,t) = \frac{1}{2\pi} \int_{B_R \cap D^*_t(x)} \sqrt{t^2 - |y-x|^2} \frac{\mu_1(y)}{4\pi} dA_y
\]

\[
(iii) \quad n = 2 : \text{support} (\mu(\cdot,t_0)) = \{ x \in \mathbb{R}^2 \mid 1x1 \leq t_0 + R \}
\]

\[
(iii) \quad n = 3 : \text{support} (\mu(\cdot,t_0)) = \{ x \in \mathbb{R}^3 \mid t_0 - R \leq 1x1 \leq t_0 + R \}
\]